Lectures 27, Tu., Nov. 21

Reading homework: Chapter 7 of reference 1

1. Basic reproduction number. The most important value in a mathematical epidemiology model is the so-called basic reproduction number (also called by many other names) R_0. There are often several way to compute it. According to Wikipedia (http://en.wikipedia.org/wiki/Basic_reproductive_rate), the basic reproduction number of an infection is the mean number of secondary cases a typical single infected case will cause in a population with no immunity to the disease in the absence of interventions to control the infection. Simply put, *it is the number of infection produced by an infected individual in its infectious period.* Following a standard procedure that we will describe below, this value can be computed according to this definition. It is useful because it helps determine whether or not an infectious disease will spread through a population. It was originally used by George MacDonald in 1952, who constructed population models of the spread of malaria. In general, $R_0 < 1$ ensures that the infection will die out while $R_0 > 1$ suggests there is some possibility of an epidemic. Mathematically, $R_0 > 1$ is often the condition that an endemic equilibrium E^* exist (with positive value for the infected class). This condition often is identical to the condition that ensures the disease free equilibrium to be unstable. We will use the second problem in the homework set 5 to illustrate these different methods.

Example Consider the model

$$\frac{dS}{dt} = \mu N - \beta S \frac{I}{N} - \mu S + \delta I \tag{1.1}$$

$$\frac{dE}{dt} = \beta S \frac{I}{N} - (\mu + \gamma) E \tag{1.2}$$

$$\frac{dI}{dt} = \gamma E - (\mu + \delta) I \tag{1.3}$$

The model and its assumptions as well as the meaning of each parameter are self evident from the model flow diagram. Here the E class describes the exposed class that may or may not develop the disease later on. Observe that the total population again stays constant ($N = S + E + I = N(0)$). Using the methods and similar arguments introduced in the previous lecture, we can show that solutions of (1.1) are positive for $t > 0$.

First, we would like to find the expression of R_0 according to the definition. In models without the exposed class, the task is often straightforward. Often it is obtained by setting $dI/dt > 0$ at $t = 0$. Biologically we can express R_0 as the product of the infectious period (I_T) with the infection rate. The infection rate is usually the effective contact rate (C_e) which is the product of the contact rate (C) with the effective rate (e). Since the rate of exiting the I class is $\mu + \delta$, we may think that the infectious period (mathematically, this is not correct!) is simply the reciprocal of this rate. Hence $I_T = 1/(\mu + \delta)$. An infected individual make β number of contact in a unit of time at the onset of the infectious disease (we assume that $S(0)$ equals to N). Since for each $\mu + \gamma$ number of exposed individual exiting the exposed class, γ number of them entering the infected class, we see that $e = \gamma/(\mu + \gamma)$. This gives that

$$R_0 = \frac{\beta\gamma}{(\mu + \delta)(\mu + \gamma)}. \tag{1.4}$$

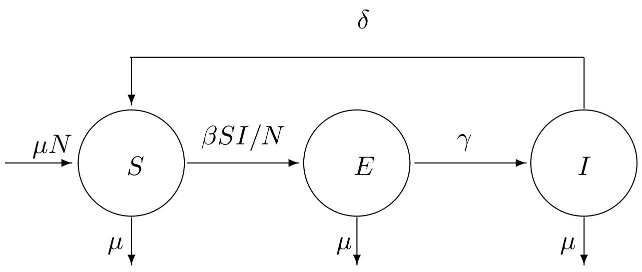

FIG. 1.1. *Flow diagram for the model (1.1)*

Taking advantage of the fact that the total population is constant, we can easily show that an endemic equilibrium, when exist, takes the form of

$$E^* = (\frac{N(0)}{R_0}, I^*, \frac{(\mu+\delta)I^*}{\gamma})$$

where

$$I^* = \frac{\mu(1-1/R_0)N(0)}{\beta/R_0 - \delta}.$$

It is easy to see that $\beta/R_0 - \delta > 0$. Hence $I^* > 0$ if and only if that $R_0 > 1$. This is an alternative method in obtaining the basic reproduction number.

We present a third method of computing R_0 via the condition for the instability of the disease free equilibrium $E_f = (N(0), 0, 0)$. We will again set $N = N(0)$.

The Jacobian at E_f is

$$J(N(0),0,0) = \begin{pmatrix} -\mu & 0 & -\beta+\delta \\ 0 & -\mu-\gamma & \beta \\ 0 & \gamma & -\mu-\delta \end{pmatrix}.$$

Obviously, the disease free equilibrium is unstable if and only if $R_0 > 1$.