Lectures 8-9, Th-Tu., Sept. 14-19

Reading homework: Chapter 2

1. Some general global stability results. It is easy to see from intuition or examples that in general, local asymptotical stability of a steady state does not imply that this steady state is also global asymptotical stability with respect to solutions of interest. In applications, many interesting models are in the form of nonlinear difference equations or systems. Obviously, in these cases, linear stability alone does not give enough information about the steady state or the system.

In this section we shall present some general yet effective results on global asymptotical stability of certain scalar nonlinear difference equations. Specifically, we are concerned here

$$x_{n+1} = f(x_n), \qquad x_0 = a, \tag{1.1}$$

where $f(x)$ is continuous and monotone or has a single peak(hump) in the domain of interest. Such functions are often encountered in applications. Some of the results of this section will be applied to specific equations in subsequent sections. We shall assume that the equation is defined on an interval of the real line. Without lose of generality, we shall further assume that such interval is invariant for the considered equation. Such assumptions are often readily met in applications. In what follows, we denote by x_n or $x_n(a)$ the solution of (1.1).

Theorem 1.1. *Assume that I is an interval of real line and that I is invariant with respect to* (1.1)*. Assume further that* (1.1) *has a steady state $x^* \in I$ and that $|(f(x) - x^*)/(x - x^*)| \leq \alpha < 1, x \in I$, where α is a positive constant. Then this steady state x^* is unique and is globally asymptotically stable on I.*

Proof. Let $x_0 \in I$. Since I is invariant, we have $x_n \in I$ for all positive integer n. It is easy to see that

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \frac{|f(x_n) - f(x^*)|}{|x_n - x^*|} \leq \alpha.$$

We thus see that $|(x_{n+1} - x^*)/(x_n - x^*)| \leq \alpha$. This clearly indicates that $|x_n - x^*| \leq \alpha^n |x_0 - x^*|$. Hence $\lim_{n\to\infty} x_n = x^*$. This proves the theorem. ∎

Corollary 1.2. *Assume that $|f'(x)| \leq \alpha < 1, x \in I$, where α is a positive constant and I is an interval of real line. Assume further that I is invariant with respect to* (1.1)*. If* (1.1) *has a steady state $x^* \in I$. Then this steady state is unique and it is globally asymptotically stable on I.*

Proof. The proof is straightforward by noticing that

$$\frac{x_{n+1} - x^*}{x_n - x^*} = \frac{f(x_n) - f(x^*)}{x_n - x^*} = f'(z),$$

where z is in between x_n and x^*. ∎

In applications, we often encounter functions of f that has a single peak. In fact, it frequently satisfies the following property:

(i): There exists $x_M \in I$, such that $f'(x)(x - x_M) < 0$ for $x \neq x_M, x \in I$.

Clearly, x_M is the peak. The next two theorems deals with such functions. The next theorem considers the case that the peak is located at the steady state or on the right hand side of the steady state.

THEOREM 1.3. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and f satisfies property (i). If* (1.1) *has a unique steady state $x^* \in I$, such that $x^* \leq x_M$. Then this steady state is globally asymptotically stable on I.*

Proof. From the assumptions about the function $f(x)$ and the uniqueness of the steady state x^*, we see that (1): if $x^* > x_0 \in I$, then $\{x_n\}_0^\infty$ is an increasing sequence satisfying $\lim_{n\to\infty} x_n = x^*$. If $x^* = x_M$, then it is easy to see that (2): if $x_0 > x^*$, then $x_1 < x^*$. This reduces this case to case (1), and therefore, we see that x_n tends to x^* as n tends to infinity. Assume below that $x_M > x^*$. Similarly, it is easy to see that (3): if $x_0 \in [x^*, x_M]$, then $\{x_n\}_0^\infty$ is decreasing with limit x^*. Let

$$x_* = \max\{x : x \in I, f(x) = x^*\}.$$

If $x_* = x^*$, then for $x_0 > x_M$, we must have $x_1 \in (x^*, x_M)$. Hence, by (3), we see that $x_n, n > 0$ form a monotone decreasing sequence with limit again x^*. If $x_* \neq x^*$. Then for $x_0 \in [x_M, x_*]$, we see as before that $x_1 \in [x^*, x_M]$, and hence by (3), we see that $\{x_n\}_1^\infty$ is decreasing with limit x^*. If $x_0 > x_*$, we have $x_1 < x^*$, hence by (1), we obtain an increasing sequence $\{x_n\}_1^\infty$ with limit x^*. This proves the theorem. ∎

Our next theorem deals with the case when the peak is on the left side of the steady state. We need the following notation.

$$x_l = \min\{x : x \in I, f(x) = x^*\}, \qquad b = f(x_M), \qquad a = f(b).$$

We note that $a < x^* < b$ and if $x_l \neq x^*$, then $x_l < x_M < x^*$. We also need the following observation.

LEMMA 1.4. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and f satisfies property (i). If* (1.1) *has a unique steady state $x^* \in I$, such that $x^* > x_M$. Then for any $x_0 \in I$, there is a nonnegative integer $m = m(x_0)$, and $c = c(x_0), c \in [x_M, x^*]$ such that $f(x_m) = f(c)$. Moreover, $[a, b]$ is invariant for* (1.1).

Proof. Observe first that if $x_0 > x^*$, then $x_1 < x^*$. So, we may assume that $x_0 < x^*$. If $x_l \neq x^*$, then we see there is a nonnegative integer m such that $x_m \in [x_l, x^*]$. It is clear that there is a unique $c \in [x_M, x^*]$ such that $f(x_m) = f(c)$. Also, we see that $x_{m+1} \in [x^*, b]$ and $x_{m+2} \in [a, x^*]$. Indeed, for all positive integer $n > m$, we have $x_n \in [a, b]$. If $x_l = x^*$, then it is also obvious that there is a unique $c \in [x_M, x^*]$ such that $f(x_0) = f(c)$. In this case, $x_1 \in [x^*, b]$ and $x_2 \in [a, x^*]$. The invariance of $[a, b]$ is obvious. This proves the lemma. ∎

Note that the next theorem gives both necessary and sufficient conditions for the global stability of a steady state.

THEOREM 1.5. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and f satisfies property (i). Assume also that* (1.1) *has a unique steady state $x^* \in I$, such that $x^* > x_M$. Then x^* is globally asymptotically stable on I if and only if that $f(f(x)) > x$ for $x \in [x_M, x^*)$.*

Proof. It is easy to see that if $f(f(x_p)) = x_p$ for some $x_p \in [x_M, x^*)$, then $x_p, f(x_p)$ form a period two cycle. If $f(f(x)) < x$ for all $x \in [x_M, x^*)$, then we see that x^* is unstable.

From the previous lemma, we see that for any $x_0 \in I$, there exists nonnegative integer m, such that $f(x_m) = f(c)$ for some $c \in [x_M, x^*]$. So, without loss of generality, we assume below that $m = 0$. If $f(f(x)) > x$ for $x \in [x_M, x^*)$, then we see that x_{2n} is strictly increasing and bounded above by x^*. Let

$$\alpha = \lim_{n \to i} x_{2n}.$$

Then we have

$$f(f(\alpha)) = \lim_{n \to i} x_{2n+2} = \alpha.$$

Therefore, we must have $\alpha = x^*$. This also yields

$$\lim_{n \to i} x_{2n+1} = f(x^*) = x^*.$$

This concludes the proof. ∎

We shall apply the above theorems to the logistic difference equation

$$x_{n+1} = r x_n (1 - x_n)$$

and the equation which we refer as Ricker's logistic equation (Ricker (1954))

$$x_{n+1} = x_n \exp r(1 - x_n/K)$$

in the next two sections.

It is not difficult to see from the proofs of the above theorems that if function f does not satisfies property (i), but has a unique steady state x^* in I, then we still have the following result that essentially due to Huang(1988):

Theorem 1.6. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and that* (1.1) *has a unique steady state $x^* \in I$. Let*

$$x_M = \sup\{x : x \in I, x \le x^*, f(x) \ge f(s), s \le x^*\}.$$

(a):If $x_M = x^*$, then x^* is globally asymptotically stable on I
(b): If $x^* > x_M$, then x^* is globally asymptotically stable on I if and only if that $f(f(x)) > x$ for $x \in [x_M, x^*)$.

When equation (1.1) has more than one steady states on I, and f satisfies property (i), we have the following theorems.

Theorem 1.7. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and f satisfies property (i). Let $x_i \in I, i = 1, 2, ...m$ be steady states of* (1.1) *such that $x_i > x_j$ for $i > j$. If $x_m \le x_M$. Then all solutions of* (1.1) *on I are eventually monotone, i.e., for $x_0 \in I$, there exists $1 \le i_0 \le m$, such that* $\lim_{n \to i} x_n(x_0) = x_{i_0}$.

In fact, the above theorem can be strengthened by specifying the regions of attraction for each of these steady states. This can be easily done geometrically. We leave this and the proof of the above theorem to interested readers.

In case that the peak $x_M \in (x_{m-1}, x_m)$, we have the following.

Theorem 1.8. *Let I be an interval of real line. Assume that I is invariant with respect to* (1.1) *and f satisfies property (i). Let $x_i \in I, i = 1, 2, ...m$ be steady states of* (1.1) *such that $x_i > x_j$ for $i > j$. If $x_m \le x_M$,. and $f(f(x)) > x$ for $x \in [x_M, x_m)$, Then for $x_0 \in I$, there exists $1 \le i_0 \le m$, such that* $\lim_{n \to i} x_n(x_0) = x_{i_0}$.

Proof. Let $l = \min\{x : x \in I, f(x) = f(x_{m-1})\}$. If $l \ne x_{m-1}$, then we can see that $J \equiv (x_{m-1}, l)$ is invariant. If $x_{m-1} = l$, then $J \equiv \{x : x \in I, x > x_{m-1}\}$ is invariant.

In both cases, we see that the conclusion now follows that of theorem 3.1.3(considering (1.1) on J) and theorem 3.1.4.

It is not difficult to see that if the function f does not satisfy property (i), but satisfy the following:

(i): Let $x_i \in I, f(x_i) = x_i, x_i < x_j$, for $i < j$, and there is a $M \in I$, such that $f(x) \in (x_i, x_{i+1})$ for $x \in (x_i, x_{i+1}), i = 1, 2, ...m-2$, and $f'(x) < 0$ for $x > x_M, x \in I$,

then theorem 3.1.4 and 3.1.5 remain true. We leave the justification details to the readers.

2. Global results on Logistic equation. In this section we shall show that if the positive steady state $x^* = 1 - r^{-1}$ of the logistic difference equation

$$x_{n+1} = rx_n(1 - x_n), \qquad x_0 \in [0, 1] \tag{2.1}$$

is locally asymptotically stable(when $1 < r < 3$), then it is in fact is globally asymptotically stable with respect to positive solutions. Indeed, we shall show that it is globally asymptotically stable even when $r = 3$.

THEOREM 2.1. *If* $1 < r \leq 3$*, then the positive steady state* $1 - r^{-1}$ *of the logistic difference equation* (2.1) *is globally asymptotically stable with respect to positive solutions with* $x_0 \in (0, 1)$.

The above theorem follows directly from the following two lemmas. We consider first when $1 < r \leq 2$.

LEMMA 2.2. *Assume that* $1 < r \leq 2$. *Then the steady state* $1 - r^{-1}$ *of the logistic difference equation* (2.1) *is globally asymptotically stable with respect to positive solutions with* $x_0 \in (0, 1)$.

Proof. We note that the peak of $f(x) = rx(1-x)$ is at $x = 1/2$ and $f(1/2) = r/4 \leq 1/2$. Let $I = (0, 1)$, then we see that I is invariant for (2.1). Clearly, f satisfies the property (i)(see the previous section) and the peak is located at the steady state or on the right hand side of it. The conclusion of the lemma follows from that of theorem 1.3. ∎

One can also use theorem 3.1.1 to prove the above lemma. The key here is to observe that $(x_{n+1} - x^*)/(x_n - x^*) = 1 - rx_n$, and for $x_0 \in (0, 1)$, $x_n \in [\min\{x_0, x_1\}, \max\{x_0, 1/2\}]$.

Although we can only conclude from local stability analysis that x^* is locally asymptotically stable when $1 < r < 3$, the next lemma indicates that in fact it is also globally asymptotically stable for $r = 3$.

LEMMA 2.3. *Assume that* $2 < r \leq 3$. *Then the steady state* $1 - r^{-1}$ *of the logistic difference equation* (2.1) *is globally asymptotically stable with respect to positive solutions with* $x_0 \in (0, 1)$.

Proof. In this case, we note that the peak $x = 1/2$ of $f(x) = rx(1-x)$ is on the left hand side of the steady state $x^* = 1 - r^{-1}$. Again we have $I = (0, 1)$ and f satisfies the property (i). The conclusion of the theorem follows from that of theorem 1.5, if we can show that $f(f(x)) > x$ for $x \in [1/2, 1 - r^{-1})$. This is equivalent to show that

$$r^2x(1-x)[1 - rx(1-x)] > x,$$

or

$$G(x, r) \equiv r^3x(1-x)^2 - r^2(1-x) + 1 < 0,$$

for

$$(x, r) \in D \equiv [1/2, 1 - r^{-1}) \times (2, 3].$$

Notice that $G(1 - r^{-1}, r) = 0$, and

$$G(1/2, r) \equiv h(r) = r^3/8 - r^2/2 + 1.$$

Note that $h(2) = 0, h(3) = -1/8 < 0$, and $h'(r) = r(3r/8 - 1)$. This indicates that h first decreases strictly on $(2, 8/3)$, followed by strictly increasing on $[8/3, 3]$. Clearly, this shows that $h(r) < 0$, for $r \in (2, 3]$. Notice also that

$$\frac{\partial G}{\partial x} \equiv G_x(x, r) = r^3(1 - x)^2 - 2r^3x(1 - x) + r^2 = r^2[3r(x - 2/3)^2 + 1 - r/3].$$

Clearly, $G_x(x, r) \geq 0$ for $2 < r \leq 3$. $G_x(x, r) = 0$ if and only if $r = 3, x = 2/3$. However the point $(x, r) = (2/3, 3)$ is not inside the set D of consideration. This proves that $G_x(x, r) > 0$ in the set of interest D. Hence, we must have for $(x, r) \in D$,

$$G(x, r) < 0.$$

This completes the proof. ∎

3. Global results on Ricker's model. For logistic difference equation (2.1), we know that if $x_0 > 1$, then $x_1 < 0$, an absurdity if it were used for population model. As a simple nonlinear population model, the following nonlinear difference equation was proposed and studied by Ricker on stock and recruitment of fish (Ricker (1954))):

$$N_{n+1} = N_n \exp r(1 - N_n/K), \qquad x_0 > 0. \tag{3.1}$$

This is the so-called *Ricker's model*, or *Ricker's logistic equation.* However, the more comprehensive study of the above equation is documented in May(1975).

Clearly, for any positive integer n, we have $x_n > 0$ for (2.1), as long as $x_0 > 0$. Here r, K are positive constants. Notice that if $x_0 > K$, then $x_1 < x_0$, and if $x_0 < K$, then $x_1 > x_0$. For this reason, K is often called the *carrying capacity* of the environment. Notice also that we can easily scale out K by letting $x_n \equiv N_n/K$. This reduces (2.1) to

$$x_{n+1} = x_n \exp r(1 - x_n), \qquad x_0 > 0. \tag{3.2}$$

Observe that (3.2) has two steady state $0, 1$. It is trivial to see that 0 is always unstable. In this section we shall show that if the positive steady state $x^* = 1$ of the equation (3.2) is locally asymptotically stable(when $0 < r < 2$), then it is in fact is globally asymptotically stable with respect to positive solutions. Indeed, we shall show that it is globally asymptotically stable even when $r = 2$.

THEOREM 3.1. *If* $1 < r \leq 2$, *then the positive steady state* $x^* = 1$ *of the equation* (3.2) *is globally asymptotically stable with respect to positive solutions.*

As in the previous section, we divide the proof of the above theorem as two lemmas. We consider first when $0 < r \leq 1$.

LEMMA 3.2. *Assume that* $0 < r \leq 1$. *Then the steady state* $x^* = 1$ *of equation* (3.2) *is globally asymptotically stable with respect to positive solutions.*

Proof. We note that the peak of $f(x) = x \exp r(1-x)$ is at $x = r^{-1} \geq 1$. Let $I = (0, +\infty)$, then we see that I is invariant for (3.2) and (3.2) has a unique steady state $x^* = 1$ in I. Clearly, f satisfies the property (i)(see the previous section) and the peak is located at the steady state x^* or on the right hand side of it. The conclusion of the lemma follows from that of theorem 1.3. ∎

One can also use theorem 1.1 to prove the above lemma. The key here is to observe that $(x_{n+1}-x^*)/(x_n-x^*) = 1-rx_n$, and for $x_0 \in (0,1)$, $x_n \in [\min\{x_0, x_1\}, \max\{x_0, 1/2\}]$.

Although we can only conclude from local stability analysis that x^* is locally asymptotically stable when $1 < r < 3$, the next lemma indicates that in fact it is also globally asymptotically stable for $r = 3$.

LEMMA 3.3. *Assume that* $1 < r \leq 2$. *Then the steady state* $x^* = 1$ *of equation* (3.2) *is globally asymptotically stable with respect to positive solutions.*

Proof. In this case, we note that the peak $x = r^{-1}$ of $f(x) = x \exp r(1-x)$ is on the left hand side of the steady state $x^* = 1$. Again we have $I = (0, +\infty)$ and f satisfies the property (i). The conclusion of the theorem follows from that of theorem 1.5, if we can show that $f(f(x)) > x$ for $x \in [r^{-1}, 1)$. This is equivalent to show that

$$x \exp r(1-x) \exp r(1 - x \exp r(1-x)) > x,$$

or

$$G(x,r) \equiv 2 - x - x \exp r(1-x) > 0,$$

for

$$(x,r) \in D \equiv [r^{-1}, 1) \times (1, 2].$$

Observe that $G(1,r) = 0$, and

$$G(r^{-1}, r) \equiv h(r)/r = 2 - r^{-1} - r^{-1} \exp(r-1).$$

That is $h(r) = 2r - 1 - \exp(r-1)$. Note that $h(1) = 0, h(2) = (3-e) > 0$, and $h'(r) = 2 - \exp(r-1)$. This indicates that h first increases strictly on $(1, 1+\ln 2)$, followed by strictly increasing on $[1+\ln 2, 2]$. Clearly, this shows that $h(r) > 0$, for $r \in (1,2]$. Hence $G(r^{-1}, r) > 0$, for $r \in (1,2]$. Notice also that

$$\frac{\partial G}{\partial x} \equiv G_x(x,r) = -1 - \exp r(1-x) + rx \exp r(1-x).$$

We shall show below that $G_x(x,r) < 0$, for $(x,r) \in D$. We let

$$g(y) = (y-1)\exp(-y).$$

Then we see that $g'(y) = (2-y)\exp(-y) > 0$, for $y < 2$. Note that $G_x(x,r) = g(rx) \exp r - 1$, and $1 \leq rx < r$ for $(x,r) \in D$. Hence we have

$$G_x(x,r) < g(r) \exp r - 1 = r - 2 \leq 0,$$

for $(x,r) \in D$. This together with the fact that $G(1,r) = 0$ and $G(r^{-1}, r) > 0$ proves that $G(x,r) > 0$, for $(x,r) \in D$. This completes the proof. ∎