Lecture 7, Tu., Sept. 12. Hw 2 is due on Th., Sept. 28

Reading homework: Chapter 2

Derivation of Euler's equation

We consider the more generalscenario that all the vital rates are functions of continuous time. Specifically, let
$Bt)$=total birth rate,
$l(x) = l_x$ = fraction of newborn to survive to age x,
$m(x) = m_x$ = average birth rate for an individual of age x.
Then

$$B(t) = \int_0^\infty \text{birth due to parents of age} x \, dx = \int_0^\infty B(t-x)l(x)m(x)dx.$$

Assume that the population is at a stable age distribution and the intrinsic growth rate is r (the dominating eigenvalue). Then $B(t) = e^{rt}B(0)$. Hence

$$e^{rt}B(0) = \int_0^\infty e^{r(t-x)}B(0)l(x)m(x)dx.$$

Which gives us the Euler's equation

$$1 = \int_0^\infty e^{-rx}l(x)m(x)dx.$$

In discrete time, the Euler's equation takes the from of

$$1 = \sum_{x=0}^\infty e^{-rx}l_x m_x.$$

In continuous time, the net reproduction rate is

$$R_0 = \int_0^\infty l(x)m(x)dx.$$

The stable age distribution is

$$c(x) = \frac{\text{number of individuals of age} x}{\text{total number of individuals}} = \frac{B(t-x)}{\int_0^\infty B(t-x)l(x)dx}.$$

Chapter 2: Nonlinear Difference Equations

1. Stability of first order nonlinear difference equations. We consider first the scalar equation

$$x_{n+1} = f(x_n), \qquad f(x) \in C^1. \tag{1.1}$$

We say $\overline{x}$ is a steady state solution (equilibrium) of (1.1) if $\overline{x} = f(\overline{x})$.

Definition 1. *The steady state solution $\overline{x}$ is stable if for any positive constant ε, there is a δ such that $|x_0 - \overline{x}| < \delta$ implies that for all $n > 0$, $|x_n - \overline{x}| < \varepsilon$. If in addition,*

$\lim_{n\to\infty} x_n = \overline{x}$, *then we say that the steady state solution* $\overline{x}$ *is asymptotically stable.*

Notice that in the textbook, stable is actually referred as asymptotically stable. The following simple theorem is very useful. We provide its rigorous proof.

THEOREM 1. *The steady state solution* $\overline{x}$ *of (1.1) is asymptotically stable if* $|df(\overline{x})/dx| < 1$.

Proof. Since $f(x) \in C^1$ and $|df(\overline{x})/dx| < 1$, there is a $\varepsilon_1 > 0$ such that $|x_0 - \overline{x}| \le \varepsilon_1$ ensures that $|df(x_0)/dx| < 1$. Then

$$\lambda \equiv \max\{|df(x_0)/dx| : |x_0 - \overline{x}| \le \varepsilon_1\} < 1.$$

Given $\varepsilon > 0$, let $\delta = \min\{\varepsilon/2, \varepsilon_1/2\}$. Recall that by the mean value theorem, we have that

$$f(x_0) = f(\overline{x}) + f'(\xi)(x_0 - \overline{x})$$

for some ξ in between x_0 and $\overline{x}$. Since $\overline{x} = f(\overline{x})$, we have

$$|x_1 - \overline{x})| = |f(x_0) - \overline{x})| = |f'(\xi)(x_0 - \overline{x})| \le \lambda|x_0 - \overline{x}| < \delta.$$

Continue this way, we obtain that

$$|x_n - \overline{x})| \le \lambda^n|x_0 - \overline{x}| < \delta.$$

Clearly, $\lim_{n\to\infty} x_n = \overline{x}$. ∎

A simple application of this theorem to the discrete logistic equation (see example 2 on page 44) $x_{n+1} = rx_n(1 - x_n)$ yields that $\overline{x} = 1 - 1/r$ exists and is asymptotically stable if $1 < r < 3$.

2. Stability of second order nonlinear difference equations. Read sections 2.7 and 2.8. Make sure that you understand and familiar with the following result.

THEOREM 2. *The roots of* $\lambda^2 - \beta\lambda + \gamma = 0$ *satisfy* $|\lambda| < 1$ *if* $|\beta| < 1 + \gamma < 2$.

3. Stability of higher order nonlinear difference equations. Read section 2.9. Make sure that you are familiar with Jury test in the special case of third order difference equations.

THEOREM 3. *The roots of* $P(\lambda) \equiv \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ *satisfy* $|\lambda| < 1$ *if and only if that*
1): $P(1) > 0$,
2): $P(-1) < 0$,
3): $|a_3| < 1, |b_3| > |b_1|, |c_3| > |c_2|$,
where $b_3 = 1 - a_3^2, b_2 = a_1 - a_3a_2, b_1 = a_2 - a_3a_1, c_3 = b_3^2 - b_1^2, c_2 = b_3b_2 - b_1b_2$.

The Jury test is a practical presentation of the so-called Schur-Cohn criterion which can be derived easily from the much more well-known Routh-Hurwitz criterion (see page 233).