

Lectures 18, Th., Oct. 19

Reading homework: Chapter 6

The third set of homework is postponed to Tu., Oct. 24.

Limit point, set and Poincare-Bendixson Theorem

Again, we took some examples from the book of Waltman (A Second Course in Elementary Differential Equations, Dover, 2004, \$11.82 at walmart.com).

Omega ($\omega(x)$) limit set (Definition)

Let $x(t) = \Phi(t, x)$ ($x(0) = x$) be the flow of the differential equation $x' = f(x)$, where $f \in C^1(M, \mathbb{R}^n)$, with M an open subset of \mathbb{R}^n . Consider $x \in M$.

The omega limit set of x , denoted $\omega(x)$, is the set of points $y \in M$ such that there exists a sequence $t_n \rightarrow \infty$ with $\Phi(t_n, x) \rightarrow y$.

Similarly, the alpha limit set of x , denoted $\alpha(x)$, is the set of points $y \in M$ such that there exists a sequence $t_n \rightarrow -\infty$ with $\Phi(t_n, x) \rightarrow y$.

Poincare-Bendixson theorem (general form, different from the one stated in the lecture)

Let M be an open subset of \mathbb{R}^2 , and $f \in C^1(M, \mathbb{R}^2)$. Consider the planar differential equation $x' = f(x)$. Consider a fixed $x \in M$. Suppose that the omega limit set $\omega(x) \neq \emptyset$ is compact, connected, and contains only finitely many equilibria. Then one of the following holds:

1. $\omega(x)$ is an equilibrium.
2. $\omega(x)$ is a periodic orbit.
3. $\omega(x)$ consists of (finitely many) equilibria $\{x_j\}$ and non-closed heteroclinic orbits (see page 193) $\gamma(y)$ such that $\omega(y) \in \{x_j\}$ and $\alpha(y) \in \{x_j\}$.

The same result holds when replacing omega limit sets by alpha limit sets.

Since f was chosen such that existence and uniqueness hold, and that the system is planar, the Jordan curve theorem implies that it is not possible for orbits of the system satisfying the hypotheses to have complicated behaviors. Typical use of this theorem is to prove that an equilibrium is globally asymptotically stable after using a Dulac type result or an ad hoc argument (like the function $Z = N + \alpha_1 C$ for the chemostat model) to rule out periodic orbits.