Duality Theory and Maximal Coactions

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Abstract:

For any nondegenerate coaction $\delta: A \to M(A \otimes C^*(G))$ of a locally compact group G on a C^*-algebra A, there exist natural surjections

$$A \times_\delta G \times_{\widehat{\delta}} G \xrightarrow{\Phi} A \otimes \mathcal{K}(L^2(G)) \xrightarrow{\Psi} A \times_\delta G \times_{\widehat{\delta},r} G,$$

neither of which need be an isomorphism. It is known, however, that if the canonical map $j_A: A \to M(A \times_\delta G)$ is injective (such coactions are called *normal* coactions), then Ψ is an isomorphism, giving us a useful Duality Theorem for normal coactions. When (A, G, δ) is not normal, there is always a *normalization* (A^n, G, δ^n) which is, where A^n is a quotient of A, the quotient map $\psi: A \to A^n$ is equivariant, and $\psi \times G: A \times_\delta G \to A^n \times_{\delta^n} G$ is an isomorphism.

In this talk we re-introduce the notion of *maximal coactions*, which are by definition those for which Φ is an isomorphism. Thus, if (A, G, δ) is maximal, a Duality Theorem holds for the *full* crossed product by the dual action. We also show that every coaction (A, G, δ) has a *maximalization*: a maximal coaction (A^m, G, δ^m) where A is a quotient of A^m, the quotient map $\phi: A^m \to A$ is equivariant, and $\phi \times G: A^m \times_{\delta^m} G \to A \times_\delta G$ is an isomorphism. For an arbitrary coaction (A, G, δ) we can conclude that crossed-product duality holds for some “intermediate” quotient $A \times_\delta G \times_{\widehat{\delta},\mu} G$ of the full crossed product.

This is joint work with Siegfried Echterhoff and John Quigg.

Duality Theorems for actions (B, G, β):

G abelian: dual action of $\widehat{G}$

$$B \times_{\beta} G \times_{\widehat{\beta}} \widehat{G} \cong B \otimes \mathcal{K}(L^2(G))$$

G non-abelian: dual *coaction* of G

$$B \times_{\beta} G \times_{\widehat{\beta}} \widehat{G} \cong B \otimes \mathcal{K}(L^2(G))$$

$$B \times_{\beta,r} G \times_{\widehat{\beta}^r} \widehat{G} \cong B \otimes \mathcal{K}(L^2(G))$$

A *coaction* of G on A is a nondegenerate $*$-homomorphism

$$\delta: A \to M(A \otimes C^*(G))$$

with certain properties.

There exist:

a crossed product $A \times_\delta G$

a dual *action* $\widehat{\delta}$ of G on $A \times_\delta G$

covariant representations

$$\begin{gathered} \pi: A \to B(\mathcal{H}) \\ \mu: C_0(G) \to B(\mathcal{H}) \end{gathered}$$

integrated forms

$$\pi \times \mu: A \times_\delta G \to B(\mathcal{H})$$

a universal covariant pair

$$\begin{gathered} j_A: A \to M(A \times_\delta G) \\ j_{C(G)}: C_0(G) \to M(A \times_\delta G) \end{gathered}$$

A Duality Theorem for Coactions (A, G, δ):

Theorem 1 (K, Quigg [2])

$$A \times_\delta G \times_{\widehat{\delta},r} G \cong (A/\ker j_A) \otimes \mathcal{K}(L^2(G))$$

Definition 2 *A coaction (A, G, δ) is* normal *if $j_A: A \to M(A \times_\delta G)$ is injective.*

Thus, for normal coactions δ:

$$A \times_\delta G \times_{\widehat{\delta},r} G \cong A \otimes \mathcal{K}(L^2(G))$$

Proposition 3 (Raeburn [6]; Quigg [5]) *For any coaction (A, G, δ) there is a normal coaction (A^n, G, δ^n) and a $\delta - \delta^n$ equivariant surjection $\psi: A \to A^n$ such that*

$$\psi \times G: A \times_\delta G \to A^n \times_{\delta^n} G$$

is an isomorphism.

Proposition 4 *For any coaction (A, G, δ), there exists an isomorphism Υ such that TFDC:*

$$\begin{array}{ccc} A \times_\delta G \times_{\widehat{\delta}} G & \xrightarrow{\Phi} & A \otimes \mathcal{K}(L^2(G)) \\ \Big\downarrow{\scriptstyle \Lambda} & & \Big\downarrow{\scriptstyle \psi \otimes \mathsf{id}} \\ A \times_\delta G \times_{\widehat{\delta},r} G & \xrightarrow[\Upsilon]{} & A^n \otimes \mathcal{K}(L^2(G)) \end{array}$$

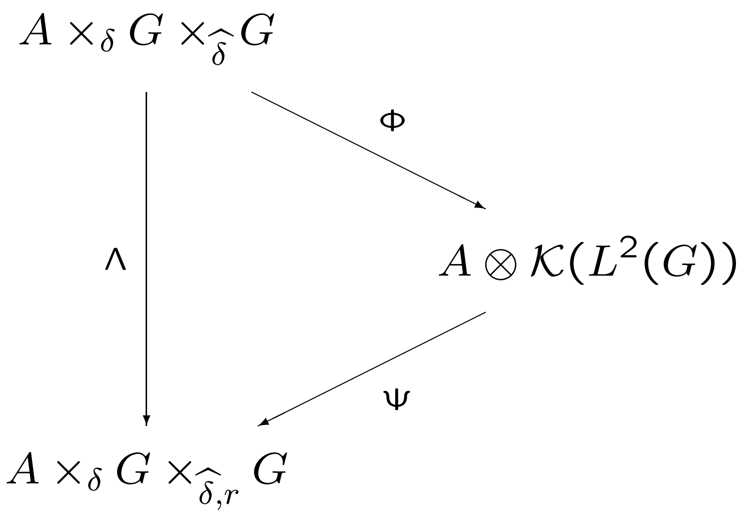
Λ: *regular representation*
Φ: *canonical surjection*

$$\Phi = ((\mathsf{id}_A \otimes \lambda) \circ \delta) \times (1 \otimes M) \times (1 \otimes \rho)$$

Proof. Since Λ, Φ, $\psi \otimes \mathsf{id}$ are all surjections, it suffices to verify that

$$\ker \Lambda = \ker(\psi \otimes \mathsf{id}) \circ \Phi.$$

Then there is in fact a unique map, Υ, such that the diagram commutes. $\diamond$



Corollary 5 *δ is normal if and only if the surjection*

$$\Psi = \Upsilon^{-1} \circ (\psi \otimes \mathrm{id})$$

is an isomorphism.

Definition 6 *A coaction (A, G, δ) is* maximal *if the canonical surjection*

$$\Phi\colon A \times_\delta G \times_{\widehat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$$

is an isomorphism.

Proposition 7 *For any action* (B, G, β)*, the dual coaction*

$$\widehat{\beta} = (i_B \otimes 1) \times (i_G \otimes u)$$

of G *on* $B \times_\beta G$ *is maximal.*

Compare with the "Full Mansfield" (cf. [4]) for dual coactions of Echterhoff, K, Raeburn [1]: for any closed normal subgroup N of G,

$$(B \times_\beta G) \times_{\widehat{\beta}} G \times_{\widehat{\widehat{\beta}}|} N \sim (B \times_\beta G) \times_{\widehat{\beta}|} G/N.$$

Taking $N = G$ we have a "Full Katayama" (cf. [3]):

$$(B \times_\beta G) \times_{\widehat{\beta}} G \times_{\widehat{\widehat{\beta}}} G \sim (B \times_\beta G).$$

Proposition 8 *If* (A, G, δ) *and* (B, G, ϵ) *are Morita equivalent coactions, then* δ *is maximal if and only if* ϵ *is maximal.*

Proof. Show that

$$\ker \Phi_A \leftrightarrow \ker \Phi_B$$

under the Rieffel correspondence between ideals set up by

$$A \times_\delta G \times_{\widehat{\delta}} G \sim B \times_\epsilon G \times_{\widehat{\epsilon}} G.$$

◇

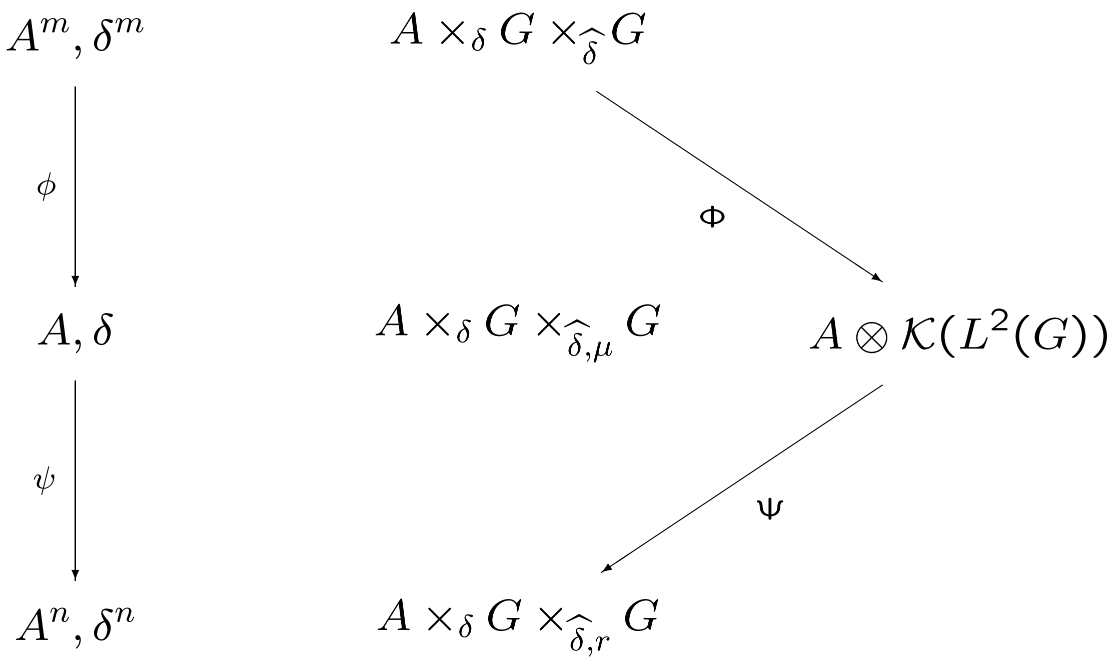
Definition 9 *Let* (A, G, δ) *be a coaction. A maximal coaction* (B, G, ϵ) *is a* maximalization *of* δ *if there exists an* $\epsilon - \delta$ *equivariant surjection* $\phi: B \to A$ *such that*

$$\psi \times G: B \times_{\epsilon} G \to A \times_{\delta} G$$

is an isomorphism.

Theorem 10 *Every coaction has a maximalization. If* (B, G, ϵ) *and* (C, G, η) *are maximalizations of* (A, G, δ) *with equivariant surjections* $\phi: B \to A$ *and* $\chi: C \to A$*, then there exists an* $\epsilon - \eta$ *equivariant isomorphism* $\theta: B \to A$ *such that* $\chi \circ \theta = \phi$*.*

Intermediate Crossed Products:



The Full Mansfield for Maximal Coactions:

(A, G, δ) maximal coaction

$N \subseteq G$ closed normal subgroup

$(B, G, \epsilon) = (A \times_\delta G \times_{\widehat{\delta}} G, G, \widehat{\widehat{\delta}})$

$$\begin{array}{ccc} A \times_\delta G \times_{\widehat{\delta|}} N & \longrightarrow & A \times_{\delta|} G/N \\ \downarrow & & \downarrow \\ B \times_\epsilon G \times_{\widehat{\epsilon|}} N & \longrightarrow & B \times_{\epsilon|} G/N \end{array}$$

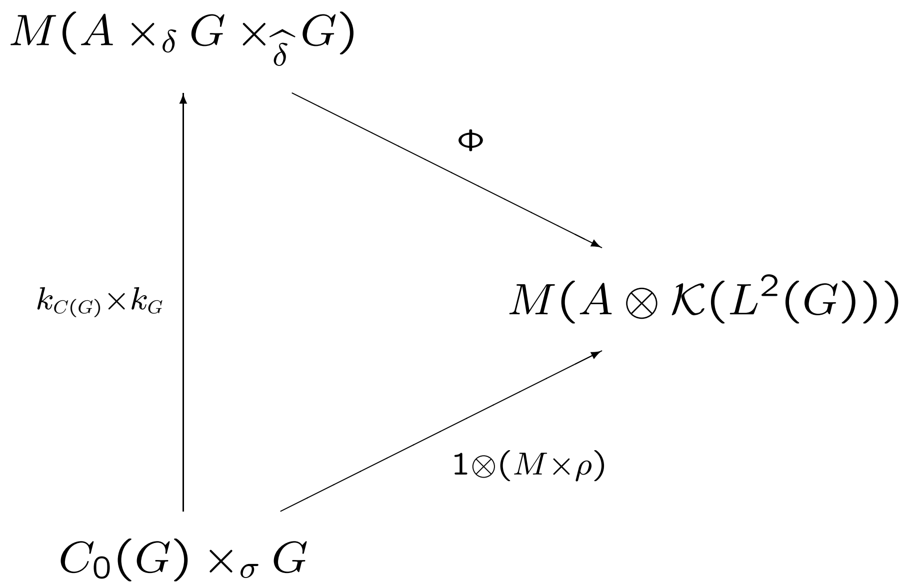
Intermediate Mansfield?

$$
\begin{array}{ccc}
A^m \times_{\delta^m} G \times_{\widehat{\delta^m|}} N & \longrightarrow & A^m \times_{\delta^m|} G/N \\
\downarrow & & \downarrow \\
A \times_{\delta} G \times_{\widehat{\delta|},\mu} N & \xrightarrow{?} & A \times_{\delta|} G/N \\
\downarrow & & \downarrow \\
A^n \times_{\delta^n} G \times_{\widehat{\delta^n|},r} N & \longrightarrow & A^n \times_{\delta^n|} G/N
\end{array}
$$

Proof. If (A^m, G, δ^m) were a maximalization of (A, G, δ), we'd have

$$A \times_\delta G \times_{\widehat{\delta}} G \cong A^m \times_{\delta^m} G \times_{\widehat{\delta^m}} G \cong A \otimes \mathcal{K}(L^2(G)).$$

Lemma 11 *For any coaction* (A, G, δ)*, TFDC:*



Put

$$A^m = p(A \times_\delta G \times_{\widehat{\delta}} G)p,$$

where P is a rank-one projection in $\mathcal{K}$ and

$$p = (k_{C(G)} \times k_G) \circ (M \times \rho)^{-1}(P).$$

Lemma 12 *The dual coaction* $\widehat{\widehat{\delta}}$ *of* G *on* $A \times_\delta G \times_{\widehat{\delta}} G$ *is exterior equivalent to a coaction* $\tilde{\delta}$ *for which* p *is invariant:*

$$\tilde{\delta}(p) = p \otimes 1.$$

In particular,

$$\delta^m = \tilde{\delta}|_{A^m}$$

is a (nondegenerate) coaction of G on A^m, which is maximal because it is Morita equivalent to a dual coaction.

Lemma 13 *The canonical surjection* Φ *is* $\tilde{\delta} - \delta \otimes_* \mathrm{id}$ *equivariant.*

Thus,

$$\phi = \Phi|_{A^m} : A^m \to A$$

is a $\delta^m - \delta$ equivariant surjection.

Thus far,

$$\begin{array}{ccc} A \times_\delta G \times_{\widehat{\delta}} G, \tilde{\delta} & \xrightarrow{\Phi} & A \otimes \mathcal{K}, \delta \otimes_* \mathsf{id} \\ {\scriptstyle p}\Big\uparrow & & \Big\uparrow{\scriptstyle 1\otimes P} \\ A^m, \delta^m & \xrightarrow{\phi} & A, \delta \end{array}$$

so

$$\begin{array}{ccc} A \times_\delta G \times_{\widehat{\delta}} G \times_{\tilde{\delta}} G & \xrightarrow{\Phi\times G} & (A \otimes \mathcal{K}) \times_{\delta \otimes_* \mathsf{id}} G \\ {\scriptstyle j(p)}\Big\uparrow & & \Big\uparrow{\scriptstyle j(1\otimes P)} \\ A^m \times_{\delta^m} G & \xrightarrow{\phi\times G} & A \times_\delta G. \end{array}$$

So $\phi \times G$ is an isomorphism if $\Phi \times G$ is.

Recall:

$$
\begin{CD}
A \times_\delta G \times_{\widehat{\delta}} G, \tilde{\delta} @>{\Phi}>> A \otimes \mathcal{K}, \delta \otimes_* \mathsf{id} \\
@V{\wedge}VV @VV{\psi \otimes \mathsf{id}}V \\
A \times_\delta G \times_{\widehat{\delta},r} G, (\tilde{\delta})^n @>{\cong}>> A^n \otimes \mathcal{K}, \delta^n \otimes_* \mathsf{id}
\end{CD}
$$

Since the vertical arrows are normalizations,

$$
\begin{CD}
A \times_\delta G \times_{\widehat{\delta}} G \times_{\tilde{\delta}} G @>{\Phi \times G}>> (A \otimes \mathcal{K}) \times_{\delta \otimes_* \mathsf{id}} G \\
@V{\cong}VV @VV{\cong}V \\
A \times_\delta G \times_{\widehat{\delta},r} G \times_{(\tilde{\delta})^n} G @>{\cong}>> (A^n \otimes \mathcal{K}) \times_{\delta^n \otimes_* \mathsf{id}} G
\end{CD}
$$

$\diamond$

References:

[1] S. Echterhoff, S. Kaliszewski, and I. Raeburn, *Crossed products by dual coactions of groups and homogeneous spaces*, J. Operator Theory **39** (1998), 151–176.

[2] S. Kaliszewski and J. Quigg, *Imprimitivity for C^*-coactions of non-amenable groups*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 101–118.

[3] Y. Katayama, *Takesaki's duality for a non-degenerate co-action*, Math. Scand. **55** (1985), 141–151.

[4] K. Mansfield, *Induced representations of crossed products by coactions*, J. Funct. Anal. **97** (1991), 112–161.

[5] J. Quigg, *Full and reduced C^*-coactions*, Math. Proc. Camb. Phil. Soc. **116** (1995), 435–450.

[6] I. Raeburn, *On crossed products by coactions and their representation theory*, Proc. London Math. Soc. **3** (1992), 625–652.