C^*-Crossed Products, Duality, and Homogeneous Spaces

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Abstract:

The theory of crossed products of C^*-algebras by coactions of locally compact abelian groups runs parallel to that of actions: each coaction of a group corresponds to an action of the dual group. In the non-abelian case, coactions still behave like actions of a dual group, until we pass to subgroups, when the symmetry spontaneously breaks down. Specifically, actions can be restricted to non-normal subgroups, but it is not known in general how to restrict coactions to non-normal quotients, *i.e.*, homogeneous spaces.

This talk will be a survey of the surprising variety of recent approaches to this problem, a variety which includes Fell bundles, generalized fixed-point algebras, graph algebras, groupoids, and Hecke algebras.

ACTIONS

$$\alpha\colon A \to C_b(G, A) \subseteq M(A \otimes C_0(G))$$

$$A \rtimes_\alpha G = \overline{\mathsf{span}}\{i_A(a)i_G(z) \mid a \in A, z \in C^*(G)\}$$

$$A \rtimes_{\alpha,r} G = \overline{\mathsf{span}}\{(\pi \otimes M) \circ \alpha(a)(1 \otimes \lambda)(z) \mid a \in A, z \in C^*(G)\}$$
$$\subseteq B(\mathcal{H}_\pi \otimes L^2(G))$$

COACTIONS

$$\delta\colon B \to M(B \otimes C^*(G))$$

$$B \rtimes_\delta G = \overline{\mathsf{span}}\{j_B(b)j_G(f) \mid b \in B, f \in C_0(G)\}$$

$$\cong \overline{\mathsf{span}}\{(\pi \otimes \lambda) \circ \delta(b)(1 \otimes M)(f) \mid b \in B, f \in C_0(G)\}$$
$$\subseteq B(\mathcal{H}_\pi \otimes L^2(G))$$

RESTRICTION

$H \subseteq G$ closed subgroup

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & M(A \otimes C_0(G)) \\ & \searrow^{\alpha|} & \downarrow {\scriptstyle \mathrm{id}_A \otimes r} \\ & & M(A \otimes C_0(H)) \end{array}$$

$N \subseteq G$ closed *normal* subgroup

$$\begin{array}{ccc} B & \xrightarrow{\delta} & M(B \otimes C^*(G)) \\ & \searrow^{\delta|} & \downarrow {\scriptstyle \mathrm{id}_B \otimes q} \\ & & M(B \otimes C^*(G/N)) \end{array}$$

DUAL COACTIONS

For an action (A, G, α) and a closed subgroup $H \subseteq G$, Green '78:

$$(A \otimes C_0(G/H)) \rtimes_{\alpha \otimes \mathrm{lt}} G \sim A \rtimes_{\alpha|} H.$$

For $N \subseteq G$ normal,

$$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N \cong (A \otimes C_0(G/N)) \rtimes_{\alpha \otimes \mathrm{lt}} G.$$

So for $H \subseteq G$ not normal we can *define*

$$B \rtimes_{\delta|} G/H := (A \otimes C_0(G/H)) \rtimes_{\alpha \otimes \mathrm{lt}} G$$

when $(B, G, \delta) = (A \rtimes_\alpha G, G, \widehat{\alpha})$ is a dual coaction.

Green's Theorem becomes:

$$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/H \sim A \rtimes_{\alpha|} H.$$

Echterhoff, K, Raeburn '98:

$$B \rtimes_\delta G \rtimes_{\widehat{\delta}} H \sim B \rtimes_{\delta|} G/H$$

for a dual coaction (B, G, δ).

DISCRETE GROUPS

For G discrete:

coaction of G	Fell bundle over G	cross-sectional algebra

$$(B, G, \delta) \longrightarrow \mathcal{B} \longrightarrow (C^*(\mathcal{B}), G, \delta^m)$$

$\mathcal{B} = \{B_s \mid s \in G\}$, $B_s = \{b \in B \mid \delta(b) = b \otimes s\}$

For $N \subseteq G$ normal (and δ maximal),

$$B \rtimes_{\delta|} G/N \cong C^*(\mathcal{B} \times G/N),$$

where $\mathcal{B} \times G/N$ is a Fell bundle over the transformation groupoid $G \times G/N$.

So for $H \subseteq G$ not normal we can *define*

$$B \rtimes_{\delta|} G/H := C^*(\mathcal{B} \times G/H)$$

when G is discrete (and is δ maximal).

Echterhoff, Quigg '02:

$$B \rtimes_{\delta} G \rtimes_{\widehat{\delta}|} H \sim C^*(\mathcal{B} \times G/H).$$

FIXED-POINT ALGEBRAS

For a reduced coaction (B, G, δ) and $N \subseteq G$ normal,

$$B \rtimes_{\delta|} G/N \cong \overline{\mathsf{span}}\{(\pi \otimes \mathsf{id}) \circ \delta(b)(1 \otimes M|(f)) \mid b \in B, f \in C_0(G/N)\} \subseteq B(\mathcal{H}_\pi \otimes L^2(G)).$$

So for $H \subseteq G$ not normal we can *define*

$$B \rtimes_{\delta,r} G/H := \overline{\mathsf{span}}\{(\pi \otimes \mathsf{id}) \circ \delta(b)(1 \otimes M|(f)) \mid b \in B, f \in C_0(G/H)\}.$$

an Huef, Raeburn p'02: The restriction to H of the dual action on $B \rtimes_\delta G$ is proper and saturated. . . Thus,

$$B \rtimes_\delta G \rtimes_{\widehat{\delta},r} H \sim (B \rtimes_\delta G)^{\widehat{\delta}|_H}.$$

Moreover,

$$(B \rtimes_\delta G)^{\widehat{\delta}|_H} \cong B \rtimes_{\delta,r} G/H.$$

GRAPH ALGEBRAS

labelled graph | skew product graph

$$\begin{array}{ccc}
(E,G,c) & \longrightarrow & E\times_c G \\
\Big\downarrow & & \Big\downarrow \\
 & & C^*(E\times_c G) \\
 & & \Big\downarrow \cong \\
(C^*(E),G,\delta_c) & \longrightarrow & C^*(E)\rtimes_{\delta_c} G
\end{array}$$

For $H \subseteq G$ any subgroup:

labelled graph | relative skew product graph

$$\begin{array}{ccc}
(E,G,c) & \longrightarrow & E\times_c G/H \\
\Big\downarrow & & \Big\downarrow \\
 & & C^*(E\times_c G/H) \\
 & & \Big\downarrow \cong \\
(C^*(E),G,\delta_c) & \longrightarrow & C^*(E)\rtimes_{\delta_c,r} G/H
\end{array}$$

MAXIMAL COACTIONS

For any coaction (B, G, δ), the map

$$\Phi := (\mathrm{id} \otimes \lambda) \circ \delta \times (1 \otimes M) \times (1 \otimes \rho) \colon \\ B \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to B \otimes \mathcal{K}(L^2(G))$$

is a surjection.

The resulting right-Hilbert bimodule:

$$B \rtimes_\delta G \rtimes_{\widehat{\delta}} G \xrightarrow{K := B \otimes L^2(G)} B$$

is called the *Katayama bimodule*.

Def'n. (B, G, δ) is *maximal* if Φ is an isomorphism.

Thus K is a $B \rtimes_\delta G \rtimes_{\widehat{\delta}} G - B$ imprimitivity bimodule when δ is maximal.

(B, G, δ) is *normal* if the kernel of Φ is the regular kernel; thus $B \otimes L^2(G)$ becomes a $B \rtimes_\delta G \rtimes_{\widehat{\delta},r} G - B$ imprimitivity bimodule when δ is normal.

Ex. Every dual coaction is maximal.

Thm. [cf. *Echterhoff, K, Quigg p'01*] *For every coaction* (B, G, δ) *there exists a maximal coaction* (B^m, G, δ^m) *and a normal coaction* (B^n, G, δ^n)*, and equivariant surjections*

$$B^m \xrightarrow{\phi} B \xrightarrow{\psi} B^n$$

which induce isomorphisms of the crossed products:

$$B^m \rtimes_{\delta^m} G \xrightarrow{\cong} B \rtimes_\delta G \xrightarrow{\cong} B^n \rtimes_{\delta^n} G.$$

$$\begin{array}{lll}
B^m & B \rtimes_\delta G \rtimes_{\widehat{\delta}} G \sim B^m & \text{“full Katayama duality”} \\
\Big\downarrow{\scriptstyle \phi} & & \\
B & B \rtimes_\delta G \rtimes_{\widehat{\delta},\mu} G \sim B & \\
\Big\downarrow{\scriptstyle \psi} & & \\
B^n & B \rtimes_\delta G \rtimes_{\widehat{\delta},r} G \sim B^n & \text{“reduced Katayama duality”}
\end{array}$$

For any coaction (B, G, δ), there exists a $\hat{\hat{\hat{\delta}}} - \delta$ compatible coaction δ_K on the Katayama bimodule.

Thus:

Thm. [*K, Quigg*] *For any maximal coaction* (B, G, δ), *and any closed normal subgroup* $N \subseteq G$,

$$B \rtimes_\delta G \rtimes_{\hat{\delta}|} N \sim B \rtimes_{\delta|} G/N.$$

Proof. Compose bimodules:

$$\begin{array}{ccc}
B \rtimes_\delta G \rtimes_{\hat{\delta}|} N & \longrightarrow & B \rtimes_{\delta|} G/N \\
{\scriptstyle K \rtimes G \rtimes N}\Big\uparrow & & \Big\uparrow{\scriptstyle K \rtimes G/N} \\
B \rtimes_\delta G \rtimes_{\hat{\delta}} G \rtimes_{\hat{\hat{\delta}}} G \rtimes_{\hat{\hat{\delta}}|} N & \xrightarrow[\text{EKR}]{} & B \rtimes_\delta G \rtimes_{\hat{\delta}} G \rtimes_{\hat{\hat{\delta}}|} G/N
\end{array}$$

□