Extensions of representations of C^*-dynamical systems

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Abstract:

Given a C^*-dynamical system (A, G, α), a closed subgroup H of G, and a covariant representation (π, U) of (A, H, α), we consider the question: For which closed subgroups K of G does there exist a representation V such that (π, V) is covariant for (A, K, α) and such that V is an extension of U? For normal H, we use non-abelian duality give an answer in terms of the induced representation $\text{Ind}_H^G(\pi \times U)$. For non-normal H, and for the related question involving covariant representations of coactions, several interesting issues arise which are related to crossed products by coactions of homogeneous spaces.

This is a preliminary report on work with Astrid an Huef, Iain Raeburn, and Dana P. Williams.

Let α be an action of a locally compact group G on a C^*-algebra A, and let H be a closed subgroup of G.

Problem. Given a covariant representation (π, U) of (A, H, α) on $\mathcal{H}$, does there exist a representation V of G on $\mathcal{H}$ such that (π, V) is covariant for (A, G, α) and $V|_H = U$?

New Problem. Given a covariant representation (π, U) of (A, H, α) on $\mathcal{H}$, *for which closed subgroups K of G* does there exist a representation V of K on $\mathcal{H}$ such that (π, V) is covariant for (A, K, α) and $V|_H = U$?

$$\begin{array}{c} \mathrm{Rep}(A \rtimes_\alpha G) \\ \Big\downarrow {\scriptstyle \mathrm{Res}} \\ \mathrm{Rep}(A \rtimes_{\alpha|} H) \end{array} \qquad\qquad \begin{array}{c} \mathrm{Rep}(A \rtimes_\alpha G) \\ \\ \mathrm{Rep}(A \rtimes_{\alpha|} K) \\ \Big\downarrow {\scriptstyle \mathrm{Res}} \\ \mathrm{Rep}(A \rtimes_{\alpha|} H) \end{array}$$

For $K = G$ and $H = N$ normal in G:

$$\begin{CD}
A \rtimes_\alpha G @= A \rtimes_\alpha G \\
@V{\text{Res}}VV @V{\text{Quasi-Reg}}VV \\
A \rtimes_{\alpha|} N @>{X_N^G\text{−Ind}}>> (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G
\end{CD}$$

X_N^G denotes Green's imprimitivity bimodule.

$$\begin{CD}
A \rtimes_\alpha G @>{Z}>> (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_\beta G/N \\
@V{\text{Quasi-Reg}}VV @V{\text{Res}}VV \\
(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G @= (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G
\end{CD}$$

Z denotes the Katayama imprimitivity bimodule for the maximal coaction $(A \rtimes_\alpha G, G/N, \widehat{\alpha}|)$; β corresponds to the dual action of G/N under the isomorphism

$$(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \cong A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N.$$

Theorem ([1])**.** Let (A, G, α) be an action, N a closed normal subgroup of G, and (π, U) a cov't rep'n of (A, N, α). Let $\rho = X_N^G$–Ind$(\pi \times U)$ be the corresponding rep'n of $B = (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G$.

There exists a rep'n V of G such that (π, V) is cov't for (A, G, α) and $V|_N = U$

if and only if

There exists a rep'n T of G/N such that (ρ, T) is cov't for $(B, G/N, \beta)$.

Proof. Combine the two previous diagrams:

$$\begin{array}{ccc}
A \rtimes_\alpha G & \xrightarrow{\;Z\;} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_\beta G/N \\
\Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\
A \rtimes_{\alpha|} N & \xrightarrow{\;X_N^G\text{–Ind}\;} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G
\end{array}$$

Prop. Let (A, G, α) be an action, N a closed normal subgroup of G, and K a closed subgroup of G with $N \subseteq K \subseteq G$. There exists an

$(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G \rtimes_{\beta|} K/N - (A \otimes C_0(G/K)) \rtimes_{\alpha\otimes \mathrm{lt}} G$

imprimitivity bimodule $Z_{G/K}^{G/N}$, and hence an inducing map

$$(A \otimes C_0(G/K)) \rtimes_{\alpha\otimes \mathrm{lt}} G \xrightarrow{\mathrm{Ind}_{G/K}^{G/N}} (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G.$$

Think of $Z_{G/K}^{G/N}$ as a Mansfield imprimitivity bimodule for full crossed products by coactions of homogeneous spaces:

$$B \rtimes_\delta Q \rtimes_{\widehat{\delta}|} R \xrightarrow{Z_{Q/R}^{Q}} B \rtimes_{\delta|} Q/R,$$

where $B = A \rtimes_\alpha G$, $Q = G/N \supseteq R = K/N$, and $Q/R = (G/N)/(K/N) \cong G/K$.

Proof. Symmetric imprimitivity theorem for certain actions of $H/N \times G$ and G on $G/N \times G$ and A.

Note that the following diagram commutes by definition:

$$
\begin{array}{ccc}
(A \otimes C_0(G/K)) \rtimes_{\alpha\otimes\mathrm{lt}} G & \xrightarrow{Z_{G/K}^{G/N}-\mathrm{Ind}} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_{\beta|} K/N \\
\Big\downarrow \mathrm{Ind}_{G/K}^{G/N} & & \Big\downarrow \mathrm{Res} \\
(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G & = & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G
\end{array}
$$

Compare with the case $K = G$:

$$
\begin{array}{ccc}
A \rtimes_\alpha G & \xrightarrow{Z} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_\beta G/N \\
\Big\downarrow \text{Quasi-Reg} & & \Big\downarrow \mathrm{Res} \\
(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G & = & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G
\end{array}
$$

Prop. With notation as above, the following diagram commutes:

$$
\begin{array}{ccc}
A \rtimes_{\alpha|} K & \xrightarrow{X_K^G\text{–Ind}} & (A \otimes C_0(G/K)) \rtimes_{\alpha\otimes \mathrm{lt}} G \\
\Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \mathrm{Ind}_{G/K}^{G/N}} \\
A \rtimes_{\alpha|} N & \xrightarrow{X_N^G\text{–Ind}} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G
\end{array}
$$

Proof. Prove directly that

$$X_N^G \otimes_{A \rtimes N} (A \rtimes_{\alpha|} K) \cong Z_{G/K}^{G/N} \otimes_{(A\otimes C_0(G/K))\rtimes G} X_K^G$$

as right-Hilbert $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G - A \rtimes_{\alpha|} K$ bimodules.

Compare with the case $K = G$:

$$
\begin{array}{ccc}
A \rtimes_{\alpha} G & = & A \rtimes_{\alpha} G \\
\Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \text{Quasi-Reg}} \\
A \rtimes_{\alpha|} N & \xrightarrow{X_N^G\text{–Ind}} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G
\end{array}
$$

Theorem (HKRW)**.** Let (A, G, α) be an action, N a closed normal subgroup of G, and (π, U) a cov't rep'n of (A, N, α). Let K be a closed subgroup of G containing N, and let $\rho = X_N^G$–Ind$(\pi \times U)$ be the corresponding rep'n of $B = (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G$.

There exists a rep'n V of G such that (π, V) is cov't for (A, K, α) and $V|_N = U$

if and only if

There exists a rep'n T of K/N such that (ρ, T) is cov't for $(B, K/N, \beta|)$.

Proof. Combine the two previous diagrams:

$$\begin{array}{ccc}
A \rtimes_{\alpha|} K & \xrightarrow{(X_K^G \otimes_{(\dots)} Z_{G/K}^{G/N})\text{–Ind}} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_{\beta|} K/N \\
\Big\downarrow \text{Res} & & \Big\downarrow \text{Res} \\
A \rtimes_{\alpha|} N & \xrightarrow{X_N^G\text{–Ind}} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G
\end{array}$$

Dual Problem. Let δ be a (nondegenerate full) coaction of G on a C^*-algebra B, and let M be a closed normal subgroup of G. Given a cov't rep'n (π, μ) of $(B, G/M, \delta|)$, *for which closed normal subgroups N of G* does there exist a rep'n ν of $C_0(G/N)$ such that (π, ν) is cov't for $(B, G/N, \delta|)$ and $\nu|_{C_0(G/M)} = \mu$?

$$
\begin{array}{ccc}
B \rtimes_{\delta|} G/N & \xrightarrow{Y^G_{G/N,*}-\mathsf{Ind}} & B \rtimes_\delta G \rtimes_{\widehat{\delta}|,*} N \\
\Big\downarrow{\scriptstyle \mathsf{Res}} & & \Big\downarrow{\scriptstyle \mathsf{Ind}^M_{N,*}} \\
B \rtimes_{\delta|} G/M & \xrightarrow{Y^G_{G/M,*}-\mathsf{Ind}} & B \rtimes_\delta G \rtimes_{\widehat{\delta}|,*} M
\end{array}
$$

$$
\begin{array}{ccc}
B \rtimes_\delta G \rtimes_{\widehat{\delta}|,*} N & \xrightarrow{X^M_{N,*}-\mathsf{Ind}} & ((B \rtimes_\delta G) \otimes C_0(M/N)) \rtimes_{\widehat{\delta}|\otimes \mathsf{lt},*} M \\
\Big\downarrow{\scriptstyle \mathsf{Ind}^M_{N,*}} & & \Big\downarrow{\scriptstyle \mathsf{Res}} \\
B \rtimes_\delta G \rtimes_{\widehat{\delta}|,*} M & = & B \rtimes_\delta G \rtimes_{\widehat{\delta}|,*} M
\end{array}
$$

Theorem(HKRW)**.** Let α be an action of G on A, N a closed *normal* subgroup of G, and $\pi \times_r U$ a repn' of $A \rtimes_{\alpha,r} G$.

$\pi \times_r U$ is equivalent to a rep'n induced from $A \rtimes_{\alpha,r} N$

if and only if

There exists a rep'n ϕ of $C_0(G/N)$ such that $(\pi \otimes \phi, U)$ is cov't for $(A \otimes C_0(G/N), G, \alpha \otimes \mathrm{lt})$.

$$\begin{array}{ccc} A \rtimes_{\alpha,r} N & \xrightarrow{X^G_{N,r}-\mathrm{Ind}} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt},r} G \\ \Big\downarrow{\scriptstyle \mathrm{Ind}^G_{N,r}} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\ A \rtimes_{\alpha,r} G & = & A \rtimes_{\alpha,r} G \end{array}$$

Proof. Non-abelian duality.

Remark. Not true for non-normal subgroups.

References.

[1] Astrid an Huef, S. Kaliszewski, and Iain Raeburn, *Extending representations of subgroups and the duality of induction and restriction* (2003), preprint.

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[3] George W. Mackey, *Unitary representations of group extensions. I*, Acta Math. **99** (1958), 265–311. MR 20 #4789