

Coactions of and Crossed Products by Homogeneous Spaces

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Abstract:

Actions of groups on C^* -algebras can be restricted to non-normal subgroups, but it is not known in general how to restrict coactions to non-normal quotients, *i.e.*, homogeneous spaces. In this talk I will survey, compare, and contrast some of the recent approaches to this problem, a variety which includes Fell bundles, generalized fixed-point algebras, graph algebras, groupoids, and Hecke algebras.

A - C^* -algebra

G - locally compact group

An *action* of G on A is a strongly continuous homomorphism $\alpha: G \rightarrow \text{Aut}(A)$.

Equivalently, an action is an injective nondegenerate homomorphism

$$\begin{aligned}\delta^\alpha: A &\rightarrow C_b(G, A) \subseteq M(A \otimes C_0(G)) \\ &= M(A \otimes C^*(\widehat{G})) \quad (G \text{ abelian})\end{aligned}$$

such that...

[properties which reflect α homomorphism]

Thus, an action β of \widehat{G} (abelian case) corresponds to

$$\delta^\beta: A \rightarrow M(A \otimes C^*(G)).$$

In general, a (full, nondegenerate) *coaction* of G on A is an injective nondegenerate homomorphism

$$\delta: A \rightarrow M(A \otimes C^*(G))$$

such that:

$$(i) \quad \delta(A)(1 \otimes C^*(G)) \subseteq A \otimes C^*(G)$$

(ii)

$$(\delta \otimes \text{id}_G) \circ \delta = (\text{id}_A \otimes \delta_G) \circ \delta$$

from A into $M(A \otimes C^*(G) \otimes C^*(G))$

($\delta_G: C^*(G) \rightarrow M(C^*(G) \otimes C^*(G))$ is the comultiplication)

$$(iii) \quad \overline{\delta(A)(1 \otimes C^*(G))} = A \otimes C^*(G).$$

\forall action $(A, G, \alpha) \exists$ crossed product C^* -algebra

$$A \times_{\alpha} G = \overline{i_A(A)i_G(C^*(G))}$$

with universal property, *dual coaction* $\hat{\alpha}$ of G , and a duality theorem:

$$(A \times_{\alpha} G) \times_{\hat{\alpha}} G \cong A \otimes \mathcal{K}(L^2(G)).$$

\forall coaction $(A, G, \delta) \exists$ crossed product C^* -algebra

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(... for δ "maximal").

RESTRICTION

(A, G, α) - action

$H \subseteq G$ - closed subgroup

$\alpha|_H: H \rightarrow \text{Aut}(A)$ is the *restricted action* of H on A .

(A, G, δ) - coaction

$H \subseteq G$ - closed normal subgroup

$\delta|_{G/N} = (\text{id}_A \otimes q) \circ \delta$ is the *restricted coaction* of G/N on A .

For $H \subseteq G$ not normal. . . ?

Diagram:

$$\begin{array}{ccc} A & \xrightarrow{\delta} & M(A \otimes C^*(G)) \\ & \searrow \delta| & \downarrow \text{id}_A \otimes q \\ & & M(A \otimes C^*(G/N)) \end{array}$$

Compare action case:

$$M(A \otimes C_0(G)) \rightarrow M(A \otimes C_0(H)).$$

Nonetheless, we can define a crossed product when H is not normal:

$$A \times_{\delta,r} G/H = \overline{j_A(A)j_G(C_0(G/H))} \subseteq M(A \times_{\delta} G),$$

and then, e.g.,

Thm. [EKR'98]

$$A \times_{\delta} G \times_{\hat{\delta},r} H \sim A \times_{\delta,r} G/H.$$

For dual coactions $\hat{\alpha}$ on $A \times_{\alpha} G$, we can define

$$(A \times_{\alpha} G) \times_{\hat{\alpha}|} G/H = (C_0(G/H) \otimes A) \times_{\tau \otimes \alpha} G,$$

and then, e.g.,

Thm. [Green '78]

$$A \times_{\alpha} G \times_{\hat{\alpha}|} G/H \sim A \times_{\alpha|} H.$$

SKEW PRODUCT GRAPHS

$E = (E^0, E^1, r, s)$ - directed graph

G - (discrete) group

$c: E^1 \rightarrow G$ - labelling

The *skew product graph* $E \times_c G$ has:

vertices $E^0 \times G$

edges $E^1 \times G$

range $r(e, g) = (r(e), g)$

source $s(e, g) = (s(e), c(e)g)$

Ex. $E = (\text{one vertex, one edge}), G = \mathbb{Z}, c(e) = 1.$

Thm. [KQR'01] \exists a coaction δ_c of G on $C^*(E)$ such that

$$C^*(E \times_c G) \cong C^*(E) \times_{\delta_c} G,$$

and hence ([KP'00])

$$C^*(E \times_c G) \times_{\gamma} G \cong C^*(E) \otimes \mathcal{K}.$$

Proof. If $C^*(E) = C^*(s_e, p_v)$, let

$$\delta_c(s_e) = s_e \otimes c(e) \quad \text{and} \quad \delta_c(p_v) = p_v \otimes 1.$$

□

RESTRICTION

(E, G, c) - labelled graph
 $H \subseteq G$ - subgroup

The *relative skew product* $E \times_c G/H$ has:

vertices $E^0 \times G/H$
edges $E^1 \times G/H$
range $r(e, g) = (r(e), gH)$
source $s(e, g) = (s(e), c(e)gH)$

Prop. [DPR p'02] If $p: F \rightarrow E$ is a connected covering (i.e., a surjective "local homeomorphism") then:

$\forall v \in F^0 \exists$ a labelling c of E by $G = \pi_1(E, p(v))$ such that

$$F \cong E \times_c G/H,$$

where $H = p_*\pi_1(F, v)$.

Ex. E as above
 $F =$ (two vertices, two edges).

$$G = \mathbb{Z}$$
$$H = 2\mathbb{Z}$$
$$c(e) = 1.$$

Thm. [DPR p'02] With $(C^*(E), G, \delta_c)$ as above,

$$C^*(E \times_c G/H) \cong C^*(E) \times_{\delta_c, r} G/H,$$

and hence

$$C^*(E \times_c G/H) \sim C^*(E \times_c G) \times_{\gamma|} H.$$

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