Hecke Groups and C^*-Algebras

S. Kaliszewski
Arizona State University

GPOTS, May 2002

Abstract:

This talk is an update on joint work with Magnus Landstad and John Quigg concerning Hecke C^*-algebras. In particular, we introduce the notion of a *Hecke group*. Briefly, a group of permutations of a set X is a Hecke group on X if the stability subgroups act with finite orbits. Taking the closure (in the topology of pointwise convergence on X) of a Hecke group Γ gives a locally compact completion of Γ; when (G, H) is a Hecke pair, taking $X = G/H$ in the above provides a natural context for what we've been calling the *Schlichting completion* of the pair (G, H).

Defn. Let G be a group, and let H be a subgroup of G.

(G, H) is a *Hecke pair* if HtH is finite in G/H for each $t \in G$.

The *Hecke algebra* of a Hecke pair (G, H) is

$$\mathcal{H}(G, H) = \text{span}\{\chi_{HtH} \mid t \in G\}$$

with the appropriate multiplication and involution.

Precedents:

Bost-Connes [1]
Brenken [2],
Laca-Raeburn [4],
Larsen-Raeburn [5],
Tzanev [7],
Hall [3]
⋮

Notice:

If G is *locally compact* and H is *compact open*, then

- ▷ $p = \chi_H$ is a (self-adjoint) projection in $C_c(G) \subseteq C^*(G)$

- ▷ $\mathcal{H} = pC_c(G)p$

- ▷ $\overline{\mathcal{H}} = pC^*(G)p \subseteq C^*(G)$ is a C^*-completion of $\mathcal{H}$, with

$$C^*(G)pC^*(G) \sim_{\text{MR}} \overline{\mathcal{H}}$$

In general:

- ▷ When is there an enveloping C^*-algebra $C^*(\mathcal{H})$ of $\mathcal{H}$?

- ▷ When $C^*(\mathcal{H})$ exists, what is it Morita-Rieffel equivalent to?

Defn. Let X be a set.

Map(X) is the space of all functions $X \to X$, with pointwise convergence (*i.e.* product) topology.

Per(X) is the subspace of all bijections $X \to X$, with the relative topology.

Facts:

Map(X) is a Hausdorff topological semigroup

Per(X) is a topological group.

Proof. For instance,

$$\begin{aligned}
\phi_i \to \phi \quad &\Rightarrow \quad \phi_i(t) = \phi(t) \text{ eventually for all } t \in X \\
&\Rightarrow \quad \phi_i(\phi^{-1}(s)) = \phi(\phi^{-1}(s) \text{ eventually for all } s \in X \\
&\Rightarrow \quad \phi_i^{-1}(s) = \phi^{-1}(s) \text{ eventually for all } s \in X \\
&\Rightarrow \quad \phi_i^{-1} \to \phi^{-1}.
\end{aligned}$$

□

Prop. *Let* Γ *be a subgroup of* Per(X) *such that the orbit* $\Gamma_s(t)$ *is finite for each* $s, t \in X$, *where*

$$\Gamma_s = \{\phi \in \Gamma \mid \phi(s) = s\}.$$

Then:

(i) $\overline{\Gamma} \subseteq$ Map(X) *is a topological group*

(ii) $\overline{\Gamma_s}$ *is compact open for each* $s \in X$

(iii) $\overline{\Gamma}$ *is a locally compact topological group.*

Proof. (i) Fix $\psi \in \overline{\Gamma}$. To see that ψ is onto:

Fix $s \in X$ and choose $\gamma \in \Gamma$ with $\gamma(s) = \psi(s)$.

Let

$$F = \Gamma_s(\gamma^{-1}(s)) \cup \{s\},$$

a finite set.

Choose $\phi \in \gamma$ with $\phi|_F = \psi|_F$.

Then:

$$\begin{aligned} \phi(s) = \psi(s) = \gamma(s) &\Rightarrow \phi^{-1}\gamma \in \Gamma_s \\ &\Rightarrow \phi^{-1}(s) = \phi^{-1}\gamma(\gamma^{-1}(s)) \in F \\ &\Rightarrow \psi(\phi^{-1}(s)) = \phi(\phi^{-1}(s) = s. \end{aligned}$$

(ii) For each $s \in X$:

$$\Gamma_s \subseteq \prod_{t \in X} \Gamma_s(t) \subseteq \prod_{t \in X} X = \mathrm{Map(X)},$$

hence compact by Tychonoff.

So $\overline{\Gamma_s}$ is compact.

Also,

$$\overline{\Gamma_s} = \overline{\Gamma} \cap \{\phi \in \mathrm{Map(X)} \mid \phi(\mathrm{s}) = \mathrm{s}\},$$

and the latter set is open and closed in Map(X), so $\overline{\Gamma_s}$ is open in $\overline{\Gamma}$.

(iii) Any $\overline{\Gamma}_s$ is a compact neighborhood of e in $\overline{\Gamma}$. □

Defn. Let X be a set. Any subgroup Γ of Per(X) such that $\Gamma_s(t)$ is finite for all $s, t \in X$ is called a *Hecke group on* X.

If Γ is a Hecke group on X, $\overline{\Gamma}$ is called the *Schlichting completion* of Γ.

By the above:

When Γ is Hecke on X, $(\overline{\Gamma})_s = \overline{\Gamma_s}$ is compact open for each $s \in X$, so $\overline{\Gamma}$ is also Hecke on X.

Ex. Let (G, H) be any group, subgroup pair, and define

$$\theta\colon G \to \mathrm{Per(G/H)} \quad \text{by} \quad \theta_{\mathrm{x}}(\mathrm{tH}) = \mathrm{xtH}.$$

Let $\Gamma = \theta(G)$. Then for all $sH, tH \in G/H$:

$$\Gamma_{sH} = \{\theta(x) \mid xsH = sH\} = \theta(sHs^{-1})$$

$$\Gamma_{sH}(tH) = \theta(sHs^{-1})tH = s(Hs^{-1}tH).$$

Thus:

$$\Gamma \text{ is Hecke on } G/H \Leftrightarrow (G, H) \text{ is a Hecke pair}$$

Also:

$$xtH = tH \ \forall tH \in G/H \Leftrightarrow x \in \bigcap_{t \in G} tHt^{-1}$$

so

$$G \cong \Gamma \Leftrightarrow (G, H) \text{ is reduced.}$$

Defn. Let (G, H) be a reduced Hecke pair, and identify G and H with $\theta(G)$ and $\theta(H)$ in Map(G/H).

The *Hecke topology* on G is the group topology from $G \subseteq$ Map(X).

The pair $(\overline{G}, \overline{H})$ is called the *Schlichting completion* of (G, H).

By the above:

- ▷ $\overline{G}$ is a locally compact topological group
- ▷ $\overline{H} = \overline{G_{eH}}$ is a compact open subgroup of $\overline{G}$.
- ▷ $(\overline{G}, \overline{H})$ is a reduced Hecke pair.

Also:

- ▷ $G/H \cong \overline{G}/\overline{H}$ G-equivariantly
- ▷ $\mathcal{H}(G, H) \cong \mathcal{H}(\overline{G}, \overline{H})$.

Inverse Limits

Recall that:

$$\mathsf{Map(X)} = \prod_{\mathsf{t \in X}} \mathsf{X} = \varprojlim_{\mathsf{F \subseteq X \text{ finite}}} \mathsf{Map(F, X)}$$

where for $F \supseteq E$,

$$\mathsf{Map(F, X)} \to \mathsf{Map(E, X)} \quad \text{by} \quad \phi \mapsto \phi|_{\mathsf{E}}.$$

More generally, for $S \subseteq \mathsf{Map(X)}$:

$$\varprojlim_{F \subseteq X \text{ finite}} S|_F = \{\phi \in \mathsf{Map(X)} \mid \phi|_{\mathsf{F}} \in \mathsf{S}|_{\mathsf{F}}\} = \overline{\mathsf{S}},$$

where for $F \supseteq E$,

$$S|_F = \{\phi|_F \mid \phi \in S\} \to S|_E \quad \text{by} \quad \phi|_F \mapsto \phi|_E.$$

Ex. Let X be a set, and let Γ be a subgroup of Per(X).

For any finite subset F of X,

$$\Gamma|_F \cong_{\text{homeo}} \Gamma/\Gamma_F \quad \text{by} \quad \phi|_F \mapsto \phi\Gamma_F,$$

where $\Gamma_F = \{\phi \in \Gamma \mid \gamma|_F = \text{id}\}$ is open in Γ.

Thus:

$$\overline{\Gamma} = \varprojlim_{F \subseteq X \text{ finite}} \Gamma/\Gamma_F.$$

Cor. *The Schlichting completion $(\overline{G}, \overline{H})$ of any reduced Hecke pair (G, H) is given by the topological inverse limits*

$$\overline{G} = \varprojlim_{F \subseteq X \text{ finite}} G/G_F.$$

and

$$\overline{H} = \varprojlim_{F \subseteq X \text{ finite}} H/H_F.$$

Ex. Let $G = \mathbb{Q} \rtimes \mathbb{Q}^+$, with $H = \mathbb{Z} \subseteq \mathbb{Q} \subseteq G$. Then (G, H) is a reduced Hecke pair.

Using the inverse limits, $\overline{G}$ can be identified as

$$\overline{G} = \mathcal{A} \rtimes \mathbb{Q}^+,$$

where $\mathcal{A} = \varprojlim_{x \in \mathbb{Q}^+} \mathbb{Q}/x\mathbb{Z}$ is the rational Adeles.

$\overline{H}$ can be identified as the integer Adeles:

$$\overline{H} = \mathcal{Z} = \varprojlim_{x \in \mathbb{Z}^+} \mathbb{Z}/x\mathbb{Z}.$$

Incidentally, if $N = \mathbb{Q} \subseteq G$, then $\overline{N} = \mathcal{A}$.

References:

[1] J.-B. Bost and A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) **1** (1995), 411–457.

[2] B. Brenken, *Hecke algebras and semigroup crossed product C^*-algebras*, Pacific J. Math. **187** (1999), 241–262.

[3] R. W. Hall, *Hecke C^*-algebras*, Ph.D. thesis, Pennsylvania State University, 1999.

[4] M. Laca and I. Raeburn, *A semigroup crossed product arising in number theory*, J. London Math. Soc **59** (1999), 330–344.

[5] N. S. Larsen and I. Raeburn, *Faithful representations of crossed products by $\mathbb{N}^k$*, Math. Scand., to appear.

[6] G. Schlichting, *On the periodicity of group operations*, Group theory (Singapore, 1987), de Gruyter (Berlin), 1989, pp. 507–517.

[7] K. Tzanev, *C^*-algèbres de Hecke et K-théorie*, Ph. D. thesis, Univ. Paris 7, December 2000.