Extensions of Group Representations

S. Kaliszewski
Arizona State University

AMS Joint Central and Western Section Meeting
and
ArlanFest
Boulder, October 2003

Abstract:

Let G be a locally compact group, and suppose U is a representation of a closed subgroup H of G on a Hilbert space. In this preliminary report on joint work with Astrid an Huef and Iain Raeburn, we consider the following problem: when does U extend to a representation of G on the same Hilbert space? Our approach uses crossed-product duality to convert the extension problem into a regularity problem for an induced representation; recent versions of Mansfield's imprimitivity theorem are then used to address the regularity problem.

Problem. Let G be a locally compact group and let H be a closed subgroup of G. Given a representation U of H on a Hilbert space $\mathcal{H}$, does there exist a representation V of G on $\mathcal{H}$ such that $V|_H = U$?

New Problem. Let α be an action of G on a C^*-algebra A. Given a covariant representation (π, U) of (A, H, α) on $\mathcal{H}$, does there exist a representation V of G on $\mathcal{H}$ such that (π, V) is covariant for (A, G, α) and $V|_H = U$?

$$\begin{array}{ccc} \mathrm{Rep}(G) & \qquad & \mathrm{Rep}(A \rtimes_\alpha G) \\ \Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\ \mathrm{Rep}(H) & & \mathrm{Rep}(A \rtimes_{\alpha|} H) \end{array}$$

For G abelian (Gootman '85, Echterhoff '94):

$$\begin{array}{ccc} \text{Rep}(A \rtimes_\alpha G) & \xrightarrow{\text{Reg}} & \text{Rep}(A \rtimes_\alpha G \rtimes_{\hat{\alpha}|} H^\perp) \\ {\scriptstyle \text{Res}}\downarrow & \nearrow {\scriptstyle X_H^G\text{−Ind}} & \\ \text{Rep}(A \rtimes_{\alpha|} H) & & \end{array}$$

X_H^G is Green's imprimitivity bimodule.

Reg: $\text{Rep}(B) \to \text{Rep}(B \times_\beta Q)$ is defined by

$$\rho \mapsto (\rho \otimes M) \circ \beta \times (1 \otimes \lambda)$$
$$\mathcal{H} \mapsto \mathcal{H} \otimes L^2(Q).$$

Key fact:

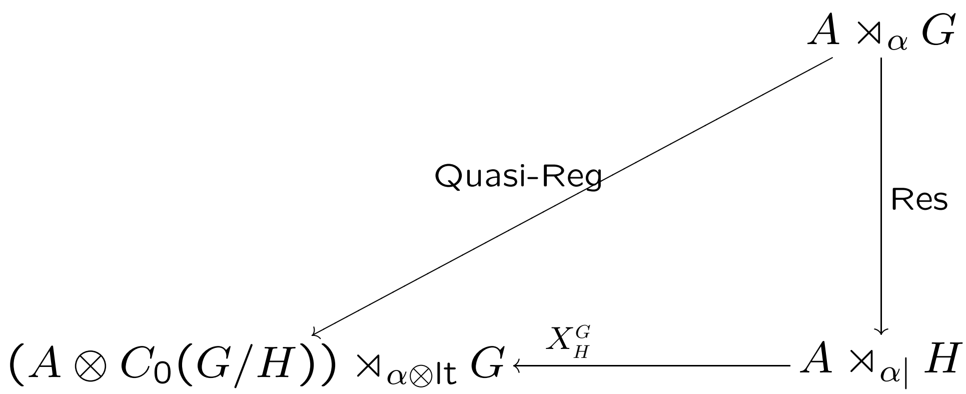
$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow{\text{Reg}} & A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \widehat{G} \\ {\scriptstyle \text{Res}}\downarrow & \nearrow {\scriptstyle X_e^G\text{−Ind}} & \\ A & & \end{array}$$

More generally, for any action (A, G, α) and closed subgroup $H \subseteq G$, let X_H^G denote Green's

$$(A \otimes C_0(G/H)) \rtimes_{\alpha \otimes \mathrm{lt}} G - A \rtimes_{\alpha|} H$$

imprimitivity bimodule.

Lemma 1.



Here the *quasi-regular rep'n* of $(A \otimes C_0(G/H)) \rtimes_{\alpha \otimes \mathrm{lt}} G$ induced from a rep'n $\phi \times W$ of $A \rtimes_\alpha G$ on $\mathcal{H}$ is

$$(\phi \otimes M) \times (W \otimes \lambda^{G/H})$$

on $\mathcal{H} \otimes L^2(G/H)$.

For $H = N$ normal in G, there is a coaction formulation:

$$\begin{array}{ccc} & & A \rtimes_\alpha G \\ & \swarrow^{\text{Reg}} & \downarrow \text{Res} \\ A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N & \xleftarrow{X_N^G} & A \rtimes_{\alpha|} N \end{array}$$

The *regular rep'n* of $A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N$ induced from a rep'n $\phi \times W$ of $A \rtimes_\alpha G$ on $\mathcal{H}$ is

$$((\phi \times W) \otimes \lambda^{G/N}) \circ \widehat{\alpha} \times (1 \otimes M)$$

on $\mathcal{H} \otimes L^2(G/N)$.

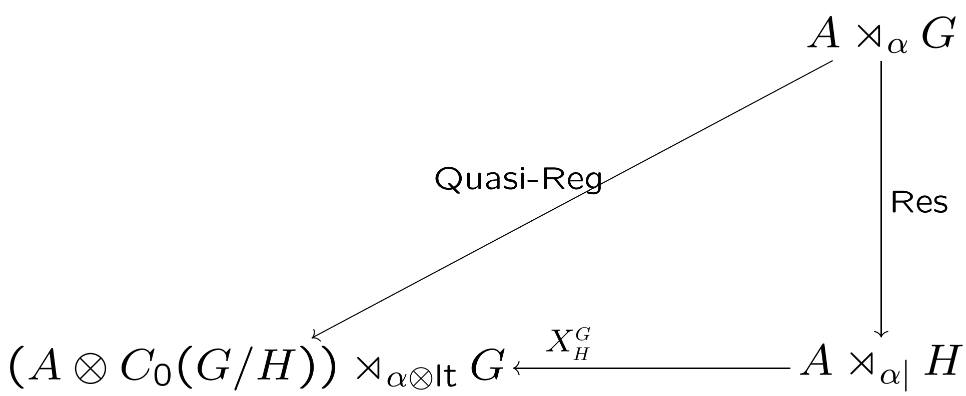
Theorem. Let (A, G, α) be an action, H a closed subgroup of G, and (π, U) a cov't rep'n of $A \rtimes_{\alpha|} H$.

There exists a rep'n V of G such that (π, V) is cov't for (A, G, α) and $V|_H = U$

if and only if

X_H^G–Ind$(\pi \times U)$ is equivalent to a quasi-regular rep'n of $(A \otimes C_0(G/H)) \rtimes_{\alpha \otimes \mathrm{lt}} G$.

Proof. Use Lemma 1 and the Rieffel correspondence:



Recall: for any action (B, Q, β),

$$\begin{array}{ccc}
(B \otimes C_0(Q)) \rtimes_{\beta \otimes \mathrm{lt}} Q & \xleftarrow{X_e^Q} & B \\
\Big\downarrow \text{Res} & \swarrow_{\text{Reg}} & \\
B \rtimes_\beta Q & &
\end{array}$$

Dually, for any *coaction* (B, Q, δ),

Lemma 2.

$$\begin{array}{ccc}
B \rtimes_\delta Q \rtimes_{\widehat{\delta}} Q & \xleftarrow{Z} & B \\
\Big\downarrow \text{Res} & \swarrow_{\text{Reg}} & \\
B \rtimes_\delta Q & &
\end{array}$$

Here $Z = B \otimes L^2(Q)$ is the *Katayama bimodule*, which is a right-Hilbert bimodule with the left action from the *canonical surjection*

$$\Phi = (\mathrm{id}_B \otimes \lambda) \circ \delta \times (1 \otimes M) \times (1 \otimes \rho)$$

of $B \rtimes_\delta Q \rtimes_{\widehat{\delta}} Q$ onto $B \otimes \mathcal{K}(L^2(Q))$.

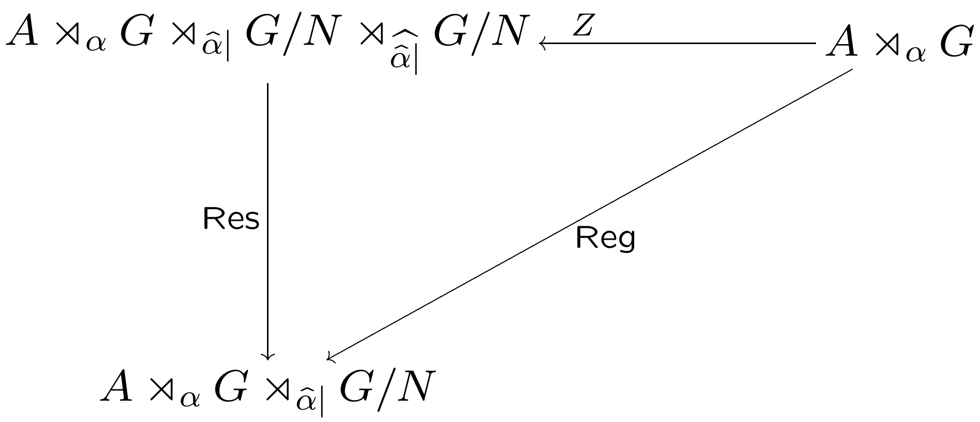
Theorem. Let (A, G, α) be an action, let N be a closed normal subgroup of G, and let ρ be a representation of $A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N$ on $\mathcal{H}$.

ρ is equivalent to a regular rep'n

if and only if

there exists a rep'n T of G/N on $\mathcal{H}$ such that (ρ, T) is cov't for $(A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N, G/N, \widehat{\widehat{\alpha}|})$.

Proof. Apply Lemma 2 to the coaction $(A \rtimes_\alpha G, G/N, \widehat{\alpha}|)$. (Since $\widehat{\alpha}|$ is *maximal*, Z is an imprimitivity bimodule.)



Lemma 3. Let (A, G, α) be an action, H a closed subgroup of G, and N a closed normal subgroup of G with $N \subseteq H \subseteq G$. There exists an

$(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G \rtimes_\beta H/N - (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes \mathrm{lt}} G$

imprimitivity bimodule Y.

Think:

$$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/N \rtimes_{\widehat{\widehat{\alpha}|}|} H/N - A \rtimes_\alpha G \rtimes_{\widehat{\alpha}|} G/H$$

or

$$B \rtimes_\delta Q \rtimes_{\widehat{\delta}|} R - B \rtimes_{\delta|} Q/R.$$

Theorem. Let ρ be a rep'n of $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes \mathrm{lt}} G$ on $\mathcal{H}$, and let Y–Ind$(\rho) = \mu \times T$.

ρ is equivalent to a quasi-regular rep'n

if and only if

there exists a rep'n S of G/N on $Y \otimes \mathcal{H}$ such that (μ, S) is cov't for $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G, G/N, \beta)$ and $S|_{H/N} = T$.

Corollary. There exists a rep'n V of G such that (π, V) is cov't for (A, G, α) and $V|_H = U$

if and only if

There exists a rep'n S of G/N such that (μ, S) is cov't for $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G, G/N, \beta)$ and $S|_{H/N} = T$.

$$\begin{array}{ccc} A \rtimes_\alpha G & \qquad & B \rtimes_\beta G/N \\ \Big\downarrow \text{Res} & & \Big\downarrow \text{Res} \\ A \rtimes_{\alpha|} H & & B \rtimes_{\beta|} H/N \end{array}$$

References.

[1] Siegfried Echterhoff, *Duality of induction and restriction for abelian twisted covariant systems*, Math. Proc. Cambridge Philos. Soc. **116** (1994), 301–315. MR 96a:46119

[2] Raymond C. Fabec, *A theorem on extending representations*, Proc. Amer. Math. Soc. **75** (1979), 157–162. MR 80f:22002

[3] Elliot C. Gootman, *Abelian group actions on type* I C^**-algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory (Bușteni, 1983), Lecture Notes in Math., vol. 1132, Springer, Berlin, 1985, pp. 152–169. MR 86j:46066