Canonical Operators on Graphs

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Dedicated to our friend and mentor Giorgio Picci on the occasion of his 65th birthday

Summary. This paper studies canonical operators on finite graphs, with the aim of characterizing the toolbox of linear feedback laws available to control networked dynamical systems.

1 Introduction

There is widespread current interest in distributed control of networked systems, e.g. [4], [5], [6], [12], [13], [15]. Much of the work to date centered on linear control laws, and has taken advantage the last twenty years of development in spectral graph theory. In particular the graph Laplacian, in various incarnations, has seen use as a stabilizing feedback. The property of the Laplacian used in these works has been essentially the fact that it is the generator of a reversible continuous time ergodic Markov chain: it has one zero eigenvalue and all others are strictly positive. The study we wish to propose is broader. We wish to ask which linear feedback laws are possible for actors which must communicate on a (possibly directed) network.

The coarse grain answer to this question is: those laws which respect the network structure. The present work, in initiating this study, precisely defines and characterizes in some detail classes of canonical (di)graph operators constructed from the incidence relations. These ideas are implicit or glossed over in a number of earlier publications; we felt that there will be those readers who, like us, benefit from the careful codification of properties. Our methods have a pronounced geometric and functorial flavor. There is a literature which has also taken this perspective; see e.g. [8], [11], [19]. We then turn to characterizing the graph Laplacian as constructed

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from differences of these canonical operators using basic linear algebraic operations. This section is remarkable in that it touches only tangentially the wide and profound literature of spectral graph theory. However, building on our foundation of properties of the fundamental operators and pursuing analogies with algebraic topological and differential geometric constructs, we are able to characterize operators previously little considered in the spectral graph theory literature. These include the Laplace-deRham operator on the edge space and Dirac operators. We show that these contain much the same graph theoretic information as the Laplacian.

The next section discusses the properties of Laplacians on weighted (di)graphs, and bears considerable relationship, and some differences in perspective, with constructions of Bensoussan and Menaldi [1] and Chung [3]. We include this section because we are able to turn this machinery to a geometric context for the Laplacian of Chung [3]. Early on this operator had mystified us and we hope that this section may provide an entry point for other readers. From these we turn to characterize the properties of operators based on other combinations of the canonical operators. In the former instance, we consider properties related to the undirected incidence operator. Here we note the prior contribution of Van Nuffelen [18]. Lastly we consider complex combinations of the canonical operators. There is a mathematical physics literature which touches such objects, but there the operators may be considered Laplacians on weighted graphs with complex weights [14], [16], [17]. In our situation this does not seem to be the case. In these latter sections, some of our results recapitulate the literature, and in some instances we have not yet been able to discover close analogs of our results. The literature in these areas seems to be relatively poorly developed, perhaps there are new results. We ask indulgence of the knowledgeable reader to direct us to related publications.

2 Graphs

2.1 The Geometry of Graphs and Digraphs

In this section we will make contact with graph theory, and describe our somewhat ideosyncratic perspective on the geometry of these objects. The objects of spectral graph theory tend to be described in terms of one of several matrices; as our perspective has been formed by contact with functorial constructions in differential geometry, operator theory and probability, our discussion will bear a marked resemblance to these areas of mathematics.

Definition 1. *1. A* digraph *(or* directed graph*) is a pair $G = (V, E)$ where V is a finite set called the* vertex set *and E is a subset of $V \times V$ called the* edge set*.*

2. *A* graph *is a pair $G = (V, E)$ where V is a finite set and E is a subset of $V \odot V$, where $\odot$ denotes the symmetric cartesian product; e.g. $V \odot V$ consists of all* unordered *pairs of elements of V, i.e. equivalence classes of the relation $(u, v) \sim (v, u)$ on $V \times V$.*
3. *A* multi-graph *is a pair $G = (V, E)$ where V is a finite set and E is a subset of the disjoint union of an finite number of copies of $V \odot V$.*

Note that we allow self edges unless otherwise stated. A more standard and visual set of definitions are: a digraph is a set of points in which some pairs of points are connected by arrows, while a graph is a set of points connected by lines. The *order* of a graph (resp. digraph) is the number of vertices, $|V|$. The *size* of a graph (resp. digraph) is the number of edges, $|E|$. A digraph is specified by its *transition matrix* $M(G)$, which is a $|V| \times |V|$ binary-valued matrix in which the entries $m_{i,j}(G) = 1$ iff $(v_i, v_j) \in E$. The *out-degree* of a vertex $v \in V$ of a digraph is the cardinality $d_o(v) = |\{e \in E : \exists u \in V s.t.\ e = vu\}|$, the *in-degree* is the cardinality $d_i(v) = |\{e \in E : \exists u \in V s.t.\ e = uv\}|$ and the *degree* of v is $d(v) = d_o(v) + d_i(v)$. The degrees of the digraph are defined as $d_i(G) = \max_{v \in V} d_i(v)$, $d_o(G) = \max_{v \in V} d_o(v)$, $d(G) = \max_{v \in V} (d_i(v) + d_o(v))$. For a (multi)graph there is only one kind of edge, so only one kind of order. A symmetric transition matrix $M(G)$ specifies a digraph in which every edge is doubled by an edge in the opposite direction, the same data also specifies a graph subject to the understanding that $(u, v) \sim (v, u)$. The *forgetful morphism* Φ maps a digraph $G = (V, E)$ to the graph $\tilde{G} = (V, \tilde{E})$ of the same order in which each edge $e = (u, v) \in E \subset V \times V$ is mapped to its equivalence class $[u, v] \in V \odot V$.

Associated to each digraph are a canonical pair of mappings,

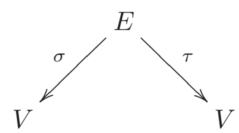
$$\sigma, \tau : E \to V,$$

the *source* map and the *target* map. To be precise:

Definition 2. *1. The* source map $\sigma : E \to V$ *is defined by* $\sigma((u, v)) = u$, *for* $(u, v) \in E$.

2. The target map $\tau : E \to V$ *is defined by* $\tau((u, v)) = v$, *for* $(u, v) \in E$.

The diagram that describes this is:



In the same way, associated to each graph is a canonical map, the incidence map:

Definition 3. *The incidence map of a graph is* $\iota : E \to 2^V$ *defined by* $\iota([u, v]) = \{u, v\} \subset V$.

Given two digraphs (resp. graphs) $G = (V, E)$ and $H = (U, F)$ a *digraph homomorphism* (resp. *graph homomorphism*) $\phi : G \to H$ is a pair of maps $\phi_V : V \to U$ and $\phi_E : E \to F$ such that the source and target maps (resp. incidence map) commute with ϕ: $\sigma\phi_E(e) = \phi_V(\sigma(e))$ and $\tau\phi_E(e) = \phi_V(\tau(e))$ (resp. $\iota\phi_E(e) = \phi_V(\iota(e))$). A digraph homomorphism is *surjective* if both ϕ_V and ϕ_E are surjective, and *injective* if both ϕ_V and ϕ_E are injective. A digraph homomorphism is *vertex surjective* (resp. *vertex injective*, resp. *edge surjective*, resp. *edge injective*) in case that ϕ_V is surjective (resp. ϕ_V is injective, resp. ϕ_E is surjective, ϕ_E is injective). A *forgetful homomorphism* of digraphs $\phi : G \to H$ is a homomorphism $\phi : \Phi G \to \Phi H$

of the associated graphs. A forgetful homomorphism maps edges to edges without specifying their direction.

We can use homomorphisms to capture properties of (di)graphs in terms of properties of simpler (di)graphs. Most important of these simpler (di)graphs are *line segments* and *circles*.

Definition 4. • *The line segment I_n is a digraph (resp. graph) in which the vertex set is the finite set of integers $\{1, 2, \cdots, n\}$, and the edges are the pairs of adjacent integers $\{(i, i+1) : i = 1, \cdots, n-1\}$ in increasing order, respectively without order.*
- *A circle is a digraph (resp. graph) in which the vertex set is $\mathbf{Z}_k$ for some k, and in which the edge set is the pairs of adjacent integers mod k in increasing order mod k, resp. the pairs of adjacent integers mod k without order.*

A *finite path* in a graph G is a forgetful homomorphism $\phi : I_k \to G$ of a line segment into G, i.e. a sequence $\{v_1, v_2, ..., v_k\}$ in V such that $\forall i$, $v_i v_{i+1} \in E$ or $v_{i+1} v_i \in E$; edges of the former type are called *sense* and the latter type are called *antisense*. A *directed path* is a homomorphism of I into G; in a directed path every edge is sense. The first and last vertices of a path are called the *starting vertex* and *ending vertex*, respectively. We call a path vertex which is not starting or ending is called an *interior vertex*. Recall from the theory of Markov chains that a directed graph is called *irreducible* if for every ordered pair $(u, v) \in V \times V$ there is a directed path for which u is the starting point and v is the endpoint. For digraphs irreducible implies only one connected component, but the converse is not true. We will call a vertex v such that $d_i(v) = 0$ (resp. $d_o(v) = 0$) *germinal* (resp. *terminal*) (more commonly these are called *source* and *sink*). A graph has no germinal or terminal verticies; in this case irreducible is equivalent to connected.

2.2 Operator Theory on Graphs and Digraphs

It is common to consider a pair of vector spaces associated to a graph;

Definition 5. *1. The* vertex space *L_V is the free linear span of V, i.e. the vector space of all real (resp. complex) valued functions defined on V.*
2. *The* edge space *L_E is the free linear span of E, i.e.the vector space of all real (resp. complex) valued functions defined on E.*
3. *If W is a subset of V or E we denote its free linear space by L_W.*
4. *The* support *of a function $f \in L_V$ (resp. $g \in L_E$) is the subset $supp(f) = \{v \in V : f(v) \neq 0\}$ (resp. $supp(g) = \{e \in E : g(e) \neq 0\}$).*

For $W \subset V$, we will regard L_W as a subset of L_V by use of the convention that functions in L_W are extended to all of V by zero. For $F \subset E$, $L_F \subset L_E$. Likewise if $supp(f) = W$ then $f \in L_W$ and if $supp(g) = F$ then $g \in L_F$.

Definition 6. *Given the constructions above, there are canonically defined linear mappings, the source and target operators, obtained as the pullback of the source* *and target maps. Namely, we may define linear maps* $S, T : L_V \to L_E$ *by* $Sf = f \circ \sigma$ *and* $Tf = f \circ \tau$*. In other words ,* $Sf(e) = f(e^-)$ *and* $Tf(e) = f(e^+)$*. where we introduce the notation* $e^+ = \tau(e)$ *and* $e^- = \sigma(e)$*.*

Of course, to do computations with specific examples of these transformations it is sometimes useful to express them as matrices. (However, we will avoid doing so for the present in order to emphasize the underlying geometric structures.)

We will need to bring into sharper focus certain types of localization of E over V. Each edge of E has a unique source. If v is not terminal, it is the source of an edge of E. Thus for every vertex, the set valued mapping $\sigma^{-1} : V \to 2^E$ may be regarded as assigning v to the subset $E^\sigma_v = \sigma^{-1}\{v\} \subset E$, which is the empty subset if v is a terminal vertex. This association may be viewed as a generalized localization of points of E over V (in that $E^\sigma_v \cap E^\sigma_u = \emptyset$ if $v \neq u$), which is a true localization over V_σ, the set of nonterminal vertices (in that $E^\sigma_v \neq \emptyset$). Similarly $E^\tau_v = \tau^{-1}\{v\} \subset E$ may be viewed as a (complementary) generalized localization of E over V by τ, which is a true localization over V_τ, the set of nongerminal vertices. Thus we may write E as a disjoint union:

$$E = \coprod_{v \in V_\sigma} E^\sigma_v = \coprod_{v \in V_\tau} E^\tau_v,$$

$$E = \coprod_{v \in V} E^\sigma_v = \coprod_{v \in V} E^\tau_v,$$

although in the latter equations some terms may be the empty set. For irreducible digraphs there are no germinal or terminal vertices, in this case $V_\sigma = V_\tau = V$.

Denote the free linear span of E^σ_v (resp. E^τ_v) by L^σ_v (resp. L^τ_v). Some of these vector spaces may be $\{0\}$. Nevertheless the set $\coprod_{v \in V} L^\sigma_v$ (resp. $\coprod_{v \in V} L^\tau_v$) may be regarded as a sort of generalized vector bundle over V, with the obvious projection map $\xi \mapsto v$ for $\xi \in L^\sigma_v$ (resp. $\xi \in L^\tau_v$). Then

$$L_E = \bigoplus_v L^\sigma_v = \bigoplus_v L^\tau_v,$$

so that L_E may be regarded as the space of sections both of these vector bundles.

Lemma 1. $ker(S) = L_{V - V_\sigma}$, $ker(T) = L_{V - V_\tau}$.

We may identify some distinguished subspaces in L_E.

Definition 7. $C_\sigma(E)$ *is the set of functions which are constant on each subset* E^σ_v*, i.e.* $C_\sigma(E) = \bigoplus_v \mathbb{C}1_{E^\sigma_v}$ *(we will restrict ourselves to consideration only of vector spaces over the complex numbers). Likewise* $C_\tau(E)$ *is the set of functions which are constant on each subset* E^τ_v*, i.e.* $C_\tau(E) = \bigoplus_v \mathbb{C}1_{E^\tau_v}$*.*

Lemma 2. $Range(S) = C_\sigma(E)$, $Range(T) = C_\tau(E)$

Now, if we give the vector spaces L_V, L_E inner products, we may consider the adjoint maps $S^*, T^* : L_E \to L_V$. Different choices of inner product give rise to different operators, which is an issue we shall consider in the following sections. Now, since L_V, L_E are free vector spaces, we consider the special cases of the dot product on these spaces, i.e. the L^2 inner products induced by the counting measures on V, respectively E. Note that for these inner products $L_v^\sigma \perp L_{v'}^\sigma$ and $L_v^\tau \perp L_{v'}^\tau$ for $v \neq v'$.

Definition 8. *$M_v^\sigma \subset L_v^\sigma$ is the subspace of functions on E_v^σ which are orthogonal to the constants. $M_v^\tau \subset L_v^\tau$ is the subspace of functions on E_v^τ which are orthogonal to the constants. $M_\sigma(E) = \bigoplus_v M_v^\sigma$. $M_\tau(E) = \bigoplus_v M_v^\tau$ (where in this context $\oplus$ denotes the orthogonal direct sum).*

Lemma 3. *$L_v^\sigma = \mathbf{C}1_{E_v^\sigma} \oplus M_v^\sigma$, $L_v^\tau = \mathbf{C}1_{E_v^\tau} \oplus M_v^\tau$, $C_\sigma(E)^\perp = M_\sigma(E)$, $C_\tau(E)^\perp = M_\tau(E)$.*

Lemma 4. *For an element $g \in L_E$,*

$$S^* g(v) = \sum_{e:e^- = v} g(e), \text{ and } T^* g(v) = \sum_{e:e^+ = v} g(e).$$

Proof. Let $1_v \in L_V$ denote the indicator function of the point $v \in V$. Then

$$S^* g(v) = \langle 1_v, S^* g\rangle = \langle S1_v, g\rangle = \sum_e S1_v(e) g(e) = \sum_e 1_v(e^-) g(e) = \sum_{e:e^- = v} g(e).$$

The case for T^* is analogous. □

Lemma 5. *We have*

1. *$ker(S^*) = C_\sigma(E)^\perp = M_\sigma(E)$, $ker(T^*) = C_\tau(E)^\perp = M_\tau(E)$.*
2. *$Range(S^*) = L_{V_\sigma}$, $Range(T^*) = L_{V_\tau}$.*

We may regard the operators S, T as canonical operators associated with the digraph, as the operators S^*, T^* are also canonically associated with the digraph and our choice of inner product on L_V and L_E. Moreover, operators constructed from algebraic combinations of these operators may also be regarded as canonical. The following results will be useful.

Proposition 1. *The operators S^*S and T^*T satisfy: $\forall f \in L_V, \forall v \in V$, $S^*Sf(v) = d_o(v) f(v)$ and $T^*Tf(v) = d_i(v) f(v)$.*

Proof. $Sf(e) = f(e^-)$, so $S^*Sf(v) = \sum_{e:e^- = v} f(e^-) = f(v) \sum_{e:e^- = v} 1$. The case for T^*T is analogous.

Corollary 1. *$\|S\|^2 = d_o(G)$ and $\|T\|^2 = d_i(G)$*

Proof.

$$\|S\|^2 = \sup_{f \in L_V} \frac{\langle Sf, Sf\rangle}{\langle f, f\rangle} = \sup_f \frac{\langle f, S^*Sf\rangle}{\langle f, f\rangle} = \sup_v d_o(v),$$

since the S^*S is diagonal. The situation for T is symmetric.

Proposition 2. *The operators S^*T and T^*S satisfy*

1. $S^*Tf(v) = \sum_{e:e^-=v} f(e^+)$
2. $T^*Sf(v) = \sum_{e:e^+=v} f(e^-)$

Proposition 3. *The operators SS^* and TT^* satisfy:*

1. $SS^*g(e) = d_o(e^-)g(e)$ *for* $g \in C_\sigma(E)$ *and* $SS^*g = 0$ *for* $g \in C_\sigma(E)^\perp$.
2. $TT^*g(e) = d_i(e^+)g(e)$ *for* $g \in C_\tau(E)$ *and* $TT^*g = 0$ *for* $g \in C_\tau(E)^\perp$.

Definition 9. *the* incidence operator *of a graph $G = (V, E)$ is $\mathcal{I}_G : L_V \to L_E$ is defined by $\mathcal{I}_G f([u, v]) = f(u) + f(v)$.*

Lemma 6. *Let G be a digraph. Then $\mathcal{I}_{\Phi G} f(e) = Sf(\tilde{e}) + Tf(\tilde{e})$, where $\tilde{e}$ is any element of $\Phi^{-1}e$.*

Remarks. Unlike the source and target operator, the incidence operator is not a pull-back, e.g. of the incidence mapping. It is more common in the literature (e.g. [2], [3]) to discuss instead with the *directed incidence operator*, discussed below, for an arbitrary choice of direction to each edge. The additional ease in using D may based on a morphism with differential geometry, as has been remarked by a number of authors ([2], [3], [10]).

3 Differences, Divergences, Laplacians and Dirac Operators

One fundamental family of operators is founded on the difference operator. A fundamental reference for this section is Bollabas [2].

Definition 10. *The* difference operator *(i.e. directed incidence operator) $D : L_V \to L_E$ is defined by $Df(e) = Tf(e) - Sf(e)$.*

Definition 11. *A* cut *is a partition of the vertex set into two pieces $V = W \cup W^c$; equivalently a cut is an indicator function $1_W \in L_V$. The* cut vector *of W is the function $g \in L_E$ such that $g(e) = 1$ if $e^+ \in W$ but $e^- \notin W$, $g(e) = -1$ if $e^- \in W$ but $e^+ \notin W$ and $g(e) = 0$ otherwise. The* cut space *is the span of all cut vectors.*

Note that the indicator functions span L_V, and that each cut vector is equal to $D1_W$ for some subset $W \subset V$. From this it is an easy step (since $\{1_{\{v\}} : v \in V\}$ spans L_V) to

Proposition 4. *The cut space is equal to $Range(D)$.*

Clearly the value of a cut vector on any self edge is zero.

Definition 12. *A* cycle *is a forgetful homomorphism of a circle into G, i.e. a path in which the endpoint vertices are equal, and a* simple cycle *is an injective forgetful homomorphism of a circle into G, i.e. a cycle in which only the endpoint vertices are repeated. By abuse of notation an edge vector $g \in L_E$ which is zero except on the edges of a simple cycle and assigns the value which assigns the value $+1$ to sense edges, -1 to antisense edges is also referred to as a simple cycle. The* cycle space *is the linear span of the simple cycles in L_E.*

Note that we might also consider the *self cycle space* linear span of the set of self cycles. For the following, denote the self cycle space by $K(G)$, the cycle space by $Z(G)$ and the cut space by $B(G)$.

Definition 13. *A* connected component *of a (di)graph is a maximal subset of V having the property that every two vertices are contained in a path. A function in L_V is called* locally constant *if $f(v) = f(u)$ for any v, u such that $vu \in E$ or $uv \in E$.*

Lemma 7. *Locally constant functions are constant on each path, hence are constant on each connected component of G. Locally constant functions are a vector subspace of L_V.*

Let $C(G)$ denote the vector space of locally constant functions in L_V. Clearly the dimension of $C(G)$ is equal the number of connected components of G.

Proposition 5. $ker(D) = C(G)$*,* $dim\,(B(G)) = Order(G) - dim\,(C(G))$*.*

Proof. Clearly $Df(e) = 0$ iff f takes the same value on both sides of e. Thus $Df = 0$ iff f is constant on every path, i.e. is locally constant.

Definition 14. *The* divergence *operator is $D^* : L_E \to L_V$, the dual operator of D.*

The following is a direct consequence of Lemma 4

Lemma 8. *The divergence operator satisfies*

$$D^*g(v) = T^*g(v) - S^*g(v) = \sum_{e:e^+=v} g(e) - \sum_{e:e^-=v} g(e)$$

Lemma 9. $ker(D^*) = B(G)^\perp$*.* $Range(D^*) = C(G)^\perp$*.*

Proof. For $g \in ker(D^*)$ and for all $f \in L_V$, $0 = \langle f, D^*g\rangle = \langle Df, g\rangle$. The result follows since $B(G) = Range(D)$. Clearly $Range(D^*) \subset C(G)^\perp$, since for $f \in C(G)$, $g \in L_E$, $\langle f, D^*g\rangle = \langle Df, g\rangle = 0$. Now suppose that $Range(D^*)$ is a proper subspace of $C(G)^\perp$. Then there exists a nonzero $f \in C(G)^\perp$ which is also in the orthogonal complement of $Range(D^*)$, so $Df \neq 0$, and for all $g \in L_E$, $0 = \langle f, D^*g\rangle = \langle Df, g\rangle$. In particular this is true for $g = Df$, which implies $0 = \langle Df, Df\rangle$, of $Df = 0$, which is a contradiction.

The proof of the following follows easily from [2], p. 53.

Proposition 6. $B(G)^\perp = Z(G) \oplus K(G)$*.*

Definition 15. *The* Laplacian *of G is the operator $\Delta = D^*D$ defined on L_V. The* Laplace-de Rham *operator of G is the operator $\square = D^*D \oplus DD^*$ defined on $L_V \oplus L_E$, where $\oplus$ denotes the orthogonal direct sum.*

Proposition 7. *1.* $\Delta = S^*S + T^*T - S^*T - T^*S$*.*
2. $ker(\Delta) = C(G)$ *and* $Range(\Delta) = C(G)^\perp$*.*
3. $0 \leq \langle f, \Delta f\rangle \leq \left(d(G) + 2\sqrt{d_o(G)d_i(G)}\right)\langle f, f\rangle \leq 2d(G)\,\langle f, f\rangle$*.*

Proof. 1. Item 1 follows from the definition of D.

2. $u \in ker(\Delta)$ iff
$$0 = \langle u, \Delta u \rangle = \langle Du, Du \rangle \,,$$
i.e. iff $u \in ker(D) = C(G)$. Since Δ is self adjoint, both $ker\Delta$ and $ker(\Delta)^{\perp}$ are finite dimensional invariant subspaces, so $\Delta | ker(\Delta)^{\perp}$ is an isomorphism.
3. Note that
$$\begin{aligned}\langle f, \Delta f \rangle &= \langle f, S^*Sf \rangle + \langle f, T^*Tf \rangle - \langle f, (S^*T + T^*S)f \rangle \\ &\leq \max_v (d_i(v) + d_o(v)) \langle f, f \rangle + 2\|Sf\|\|Tf\| \\ &= \left(d(G) + 2\sqrt{d_o(G)d_i(G)} \right) \langle f, f \rangle \\ &\leq 2d(G) \langle f, f \rangle \,,\end{aligned}$$
where the inequality follows from Theorem 1, the triangle inequality and the Cauchy-Schwartz inequality, and the succeeding equality follows from Corollary 1. Note that the latter inequality in item 3 is an equality for the case of the graph which consists of a single directed cycle of even order, and f is the function which alternates between $+1$ and -1 on successive vertices; f is an eigenfunction of Δ with eigenvalue 4, $d(G) = 2$ and $d_o(G) = d_i(G) = 1$. In particular the inequalities in item 3 are tight.

Proposition 8. *1. $DD^* = SS^* + TT^* - ST^* - TS^*$.*
2. $ker(DD^*) = B(G)^{\perp}$ and $Range(DD^*) = B(G)$.
3. The eigenvalues of DD^* are the same as those of Δ, with the same multiplicity. If $\Delta f = \lambda f$ for $\lambda \neq 0$ a constant, then Df is an eigenvector for DD^* with the same eigenvalue.

Thus the Laplace-deRham operator contains only a little additional information about the graph geometry beyond that contained in the Laplacian. The term 'Dirac operator' refers generally to a square root of the Laplacian, although by custom not to the symmetric square root of the Laplacian. Hence:

Definition 16. *The* Dirac operator *of G is the operator $\partial = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ defined on $L_V \oplus L_E$.*

Proposition 9. *1. $\partial^2 = \Box$.*
2. ∂ is self adjoint.
3. $ker(\partial) = C(G) \oplus B(G)^{\perp}$ and $Range(\partial) = C(G)^{\perp} \oplus B(G)$.
4. The eigenvalues of ∂ are the (positive and negative) square roots of those of Δ, with the same multiplicity. If $\Delta f = \lambda f$ for $\lambda \neq 0$ a constant, then the eigenvector of $\sqrt{\lambda}$ (resp. -$\sqrt{\lambda}$) is $\begin{pmatrix} \sqrt{\lambda} f \\ Df \end{pmatrix}$ (resp. $\begin{pmatrix} \sqrt{\lambda} f \\ -Df \end{pmatrix}$).

Remarks: We have taken the convention that the Laplacian is a nonnegative operator. The other common convention is, of course, that the Laplacian is nonpositive. This perspective has the net effect of replacing eigenvalues of the Laplacian and Laplace-deRham operators by their negatives, and replacing the Dirac operator by the skew-adjoint operator $\begin{pmatrix} 0 & -D^* \\ D & 0 \end{pmatrix}$, for which the eigenvalues are imaginary.

4 Operators on Weighted Graphs

Consider a function $w : E \to \mathbf{C}$ on E, and a function $\rho : V \to \mathbf{C}$ on V, which we will call *weight functions*. Although some literature considers cases in which w, rho are complex, [14], [16], [17], we will suppose that w and ρ are both positive real functions. Then the bilinear function $\langle f_1, f_2 \rangle_\rho = \sum_v \rho(v) f_1(v) f_2(v)^*$ on L_V defines an inner product. Likewise the inner product $\langle g_1, g_2 \rangle_w = \sum_e w(e) g_1(e) g_2(e)^*$ defines an inner product on L_E. Clearly there is a wide latitude of choice of weight functions, and they influence properties of the canonical operators. We may take as fundamental the definition of the operators S, T. Then the discussion of section B is valid in its entirety through Lemma 1.3 provided that orthogonality in L_E is understood to be in the weighted sense. However, Lemma 1.4 now takes the form

Lemma 10. *For a element $g \in L_E$,*

$$S^* g(v) = \frac{1}{\rho(v)} \sum_{e:e^-=v} w(e) g(e), \text{ and } T^* g(v) = \frac{1}{\rho(v)} \sum_{e:e^+=v} w(e) g(e).$$

Proof. Let $1_v \in L_V$ denote the indicator function of the point $v \in V$. Then

$$\begin{aligned} S^* g(v) = \tfrac{1}{\rho(v)} \langle 1_v, S^* g \rangle = \tfrac{1}{\rho(v)} \langle S 1_v, g \rangle &= \tfrac{1}{\rho(v)} \textstyle\sum_e w(e) S 1_v(e) g(e) \\ = \tfrac{1}{\rho(v)} \textstyle\sum_e w(e) 1_v(e^-) g(e) &= \tfrac{1}{\rho(v)} \displaystyle\sum_{e:e^-=v} w(e) g(e). \end{aligned}$$

The case for T^* is analogous. □

Lemma 1.5 is valid in the weighted case as stated. We will revise our definitions as follows.

Definition 17. *The* out-degree *of a vertex $v \in V$ of a digraph is the sum $d_o(v) = \frac{1}{\rho(v)} \sum_{e^-=v} w(e)$, the* in-degree *is the sum $d_i(v) = \frac{1}{\rho(v)} \sum_{e^+=v} w(e)$ and the* degree *of v is $d(v) = d_o(v) + d_i(v)$. The degrees of the digraph are defined as $d_i(G) = \max_{v \in V} d_i(v)$, $d_o(G) = \max_{v \in V} d_o(v)$, $d(G) = \max_{v \in V} (d_i(v) + d_o(v))$. For a graph there is only one kind of edge, so one kind of degree.*

Note that the degrees are positive real numbers, but need no longer be integers.

Proposition 10. *The operators S^*S and T^*T satisfy: $S^*Sf(v) = d_o(v) f(v)$ and $T^*Tf(v) = d_i(v) f(v)$.*

Proof. $Sf(e) = f(e^-)$, so

$$S^*Sf(v) = \frac{1}{\rho(v)} \sum_{e:e^-=v} w(e) f(e^-) = f(v) \frac{1}{\rho(v)} \sum_{e:e^-=v} w(e).$$

The case for T^*T is analogous. □.

With the above definitions of vertex degrees, the following takes the same form as in the unweighted case.

Corollary 2. $\|S\|^2 = d_o(G)$ *and* $\|T\|^2 = d_i(G)$

Proof.

$$\|S\|^2 = \sup_{f \in L_V} \frac{\langle Sf, Sf\rangle}{\langle f, f\rangle} = \sup_f \frac{\langle f, S^*Sf\rangle}{\langle f, f\rangle} = \sup_v d_o(v),$$

since the S^*S is diagonal. The situation for T is symmetric.

Proposition 11. *The operators T^*S and S^*T satisfy:*

1. $T^*Sg(v) = \frac{1}{\rho(v)} \sum_{e:e^-=v} w(e)f(e^+)$
2. $S^*Tg(v) = \frac{1}{\rho(v)} \sum_{e:e^+=v} w(e)f(e^-)$

Proposition 12. *The operators SS^* and TT^* satisfy:*

1. $SS^*g(e) = d_o(e^-)g(e)$ *for* $g \in C_\sigma(E)$ *and* $SS^*g = 0$ *for* $g \in C_\sigma(E)^\perp$.
2. $TT^*g(e) = d_i(e^+)g(e)$ *for* $g \in C_\tau(E)$ *and* $TT^*g = 0$ *for* $g \in C_\tau(E)^\perp$.

The definition of the difference operator D remains the same in this weighted situation, and it's range is still the space of cut vectors $B(G)$, and its kernel is still the space of locally constant functions $C(G)$. The orthogonal complement $B(G)^\perp$, and the divergence operator D^* are generally different, since they defined in terms of the inner product on L_E. Let W denote the operator on L_E of multiplication by the weight function w. The following lemma is immediate, given that $WK(G) = K(G)$

Lemma 11. $B(G)^\perp = W^{-1}Z(G) \oplus K(G)$.

Lemma 12. $ker(D^*) = B(G)^\perp$ *and* $Range(D^*) = C(G)^\perp$

The following theorem takes the same form as the non-weighted case.

Proposition 13.

1. $\Delta^w_\rho f(v) = (S^*S + T^*T - S^*T - T^*S)f(v)$
2. $\Delta^w_\rho f(v) = d(v)f(v) - \frac{1}{\rho(v)} \left(\sum_{e:e^-=v} w(e)f(e^+) + \sum_{e:e^+=v} w(e)f(e^-) \right)$
3. $ker(\Delta^w_\rho) = C(G)$ *and* $Range(\Delta^w_\rho) = C(G)^{\perp_\rho}$
4. $0 \le \langle f, \Delta^w_\rho f\rangle \le \left(d(G) + 2\sqrt{d_o(G)d_i(G)}\right) \langle f, f\rangle$

Of particular recent interest are weighted graph Laplacians in the case that $\rho(v) = d(v)$ and $w(e) = 1$ for all edges in the graph. In this case the Laplacian has the representation $\Delta_d f(v) = f(v) - \frac{1}{d(v)} \left(\sum_{e:e^-=v} f(e^+) + \sum_{e:e^+=v} f(e^-)\right)$, and is self adjoint on L_V with respect to the inner product $\langle f, g\rangle_d = \sum_v \overline{f}(v)g(v)d(v)$. It will be a little easier to see the self adjointness of Δ_d if we express it in a unitarily equivalent form on a different inner product space. Specifically, note that the multiplication operator $Uf(v) = \frac{1}{\sqrt{d(v)}} f(v)$ is a unitary map from the inner product space $(L_V, \langle\cdot,\cdot\rangle_1)$ to the inner product space $(L_V, \langle\cdot,\cdot\rangle_d)$. Thus Δ_d is unitarily equivalent to the operator $U^{-1}\Delta_d U = d(v)^{-1/2}\Delta d(v)^{-1/2}$ on $(L_V, \langle\cdot,\cdot\rangle_1)$. But on this inner product space self adjointness is just symmetry, and the symmetry of $d(v)^{-1/2}\Delta d(v)^{-1/2}$ is manifest. But the latter is just the Laplacian preferred by Chung [3] because its spectrum is so closely tied to graph geometry.

5 The Incidence Operator and Its Kin

Recall that the incidence operator $\mathcal{I} : L_V \to L_E$ is defined by $\mathcal{I}f(e) = Tf(e) + Sf(e)$. If a function f is in $ker(\mathcal{I})$ it must have values of equal magnitude but opposite sign at the vertices on either side of every edge. From this follows the fact that the values taken by f on a connected component of the (di)graph are determined by its value at a single vertex. Moreover, $f(v) = 0$ for v in any cycle of odd order, hence in the connected component of a cycle of odd order. This is basically everything that needs to be known about the kernel of the incidence operator.

Lemma 13. *$ker(\mathcal{I})$ is the space of functions on V which alternate sign across every edge. If G is connected, $ker(\mathcal{I})$ is zero or one dimensional according to whether G has cycles of odd order or not. In general the dimension of $ker(\mathcal{I})$ is the number of connected components without cycles of odd order.*

Remarks:. Since a connected graph is bipartite iff it has no odd cycles, $dimker(\mathcal{I})$ is the number of bipartite components. More generally the kernel contains the span of the isolated vertices. This result may be originally due to Van Nuffelen [18] in the context of graphs. A vertex with a self edge is a cycle of odd order, hence on any component containing a self edge one has $ker(\mathcal{I}) = \{0\}$.

Note that $\mathcal{I}1_v = 1_{E_v^\sigma \cup E_v^\tau}$. The set $E_v^\sigma \cup E_v^\tau$ seems to us the shadow of v on the edge set, so we will call such a vector $1_{E_v^\sigma \cup E_v^\tau}$ a *shadow*, and call a vector in the span of such vectors a shadow vector. Denote the set of shadow vectors by $\Upsilon(G)$. Clearly $Range(\mathcal{I}) = \Upsilon(G)$.

Lemma 14. *Assume G is connected. If G has cycles of odd order, $\{1_{E_v^\sigma \cup E_v^\tau} : v \in V\}$ is basis of $Range(\mathcal{I})$. Conversely, if G has no cycles of odd order, then for any $u \in V$,$\{1_{E_v^\sigma \cup E_v^\tau} : v \in V - \{u\}\}$ is a basis of $Range(\mathcal{I})$.*

Lemma 15.

$$\mathcal{I}^* g(v) = \sum_{e:e^- = v} g(e) + \sum_{e:e^+ = v} g(e).$$

Proof. This follows directly from Lemma 4.

Another way of saying the same thing, is that $\mathcal{I}^* g(v) = \left\langle 1_{E_v^\sigma \cup E_v^\tau}, g \right\rangle$. Of course $ker(\mathcal{I}^*) = Range(\mathcal{I})^\perp$. The geometry of this statement is the following. Suppose that $\mathbf{Z}_{2k}$ is a circle of even order, and that $\hat{g} \in L_E(\mathbf{Z}_{2k})$ is the alternating function: $\hat{g}\left((i, i+1)\right) = (-1)^i$, where the addition "$i + 1$" is interpreted as mod $2k$. Suppose that $c : \mathbf{Z}_{2k} \to G$ is a cycle in G. Define a function $g \in L_E$ with support contained in $c\left(\mathbf{Z}_{2k}\right)$ by $g(e) = \sum_{i:e=c((i,i+1))} \hat{g}\left((i, i+1)\right)$. Then $g \in \ker(\mathcal{I}^*)$. We will call g an *alternating cycle*. Let $A(G) \subset L_E$ denote the span of the alternating cycles. Then $A(G) \subseteq \ker(\mathcal{I}^*)$.

Definition 18. *Suppose that $g \in L_E$ and that $|supp(g) \cap (E_v^\sigma \cup E_v^\tau)| = 1$. Then we will call $e \in supp(g) \cap (E_v^\sigma \cup E_v^\tau)$ a* hanging edge.

Lemma 16. *Suppose that $\mathcal{I}^* g(v) = 0$ and that $supp(g) \cap (E_v^\sigma \cup E_v^\tau) \neq \emptyset$. Then there are at least two elements of $e, e' \in supp(g) \cap (E_v^\sigma \cup E_v^\tau)$ which satisfy $g(e)g(e') < 0$. (In other words, $supp(g)$ has no hanging edges)*

Proof. $supp(g) \cap (E_v^\sigma \cup E_v^\tau) \neq \emptyset$ implies $\sum_{e:e^-=v} |g(e)| + \sum_{e:e^+=v} |g(e)| \neq 0$. Since $\mathcal{I}^* g(v) = 0$ implies cancellation, there exists at least at least two edges $e, e' \in E_v^\sigma \cup E_v^\tau$ with $g(e) > 0$ and $g(e') < 0$. □

Definition 19. *Let $\phi : I_k \to G$ be a path, let $\hat{h}((i, i+1)) = (-1)^i$ be a function $\hat{h} \in L_E(I_k)$. Define $h \in L_E(G)$ by $h(e) = \sum_{i:e=\phi((i,i+1))} \hat{h}((i, i+1))$ if $e \in \phi(I_k)$ and $h(e) = 0$ otherwise. We will call h the* alternating path *built on ϕ. If ϕ is the restriction of a path $\phi' : I_{k+n} \to G$ for $n > 0$ and h' is the alternating path built on ϕ' we shall say that h is a* restriction *of h'.*

Lemma 17. *Assume that the alternating path h has no self edges and let v be an interior vertex of h. Then $\mathcal{I}^* h(v) = 0$.*

Proof. We have:

$$\begin{aligned}\mathcal{I}^* h(v) &= \textstyle\sum_{e:e^-=v} h(e) + \sum_{e:e^+=v} h(e) \\ &= \textstyle\sum_{e:e^-=v} \sum_{i:e=\phi((i,i+1))} \hat{h}((i, i+1)) \\ &\quad + \textstyle\sum_{e:e^+=v} \sum_{i:e=\phi((i,i+1))} \hat{h}((i, i+1)) .\end{aligned}$$

But, in the latter expression each summand is of magnitude one and uniquely paired with another such of opposite sign. Indeed, since v is an internal vertex, for every i such that $v = \phi(i)$, both edges $e = \phi((i-1, i))$ and $e' = \phi((i, i+1))$ are coincident with v, hence in the sum, while $\hat{h}((i, i+1)) = -\hat{h}((i-1, i))$. □

Lemma 18. *An alternating path of even order cannot belong to* $\ker(\mathcal{I}^*)$.

Proof. . Such a path either has a hanging edge, or in the case that the initial and terminal vertex are equal has the initial and terminal edges of equal sign, so that $\mathcal{I}^* h(v) \neq 0$ when v is the initial vertex. □

Lemma 19. *Let h be an alternating path of odd order. Then $h \in \ker(\mathcal{I}^*)$ iff $v_k = \phi(2k+1) = \phi(1) = v_1$. In other words, an alternating path is in $\ker(\mathcal{I}^*)$ iff it is of odd order and an alternating cycle.*

Proof. If $v_{2k+1} \neq v_1$ then h has a hanging edge, hence cannot be in $\ker(\mathcal{I}^*)$. Conversely, if $v_{2k+1} = v_1$ then ϕ defines a cycle of even order, and h is an alternating cycle. □

Theorem 1. *Suppose G is a multigraph. Then $ker(\mathcal{I}^*) = A(G)$.*

Remark: This result seems to be first due to Grossman et al [9], and then again by ourselves some thirteen years later.

Lemma 20. *$Range(\mathcal{I}^*)$ is the orthogonal complement of the space of alternating functions in L_V. If G has no bipartite components or isolated vertices, i.e. there is a cycle of odd order in every component, then $Range(\mathcal{I}^*)$ is all of L_V.*

Lemma 21. $\mathcal{I}^* \mathcal{I} = T^* T + S^* S + S^* T + T^* S$.

Remark: Grossman et al [9] call $\mathcal{I}^*\mathcal{I}$ the *Unoriented Laplacian* .

Proposition 14. *Suppose that G is bipartite. Then $\mathcal{I}^*\mathcal{I}$ is unitarily equivalent to Δ, the Laplacian of G.*

Proof. . Let $m \in ker\,(\mathcal{I}^*\mathcal{I})$ be real valued and unimodular (which exists by Lemma 13), and let M be the operator of multiplication by m. Then M is unitary and it's own inverse. According to Bollobas [2] p.264, $(T^*S + S^*T)M = -M(T^*S + S^*T)$. Since $T^*T + S^*S$ is a multiplication operator, it commutes with M. Thus $M^{-1}\mathcal{I}^*\mathcal{I}M = \Delta$. □

Remarks: This result is known to the algebraic graph theory community [7], although we are unaware of a specific reference. For a regular graph $\mathcal{I}^*\mathcal{I}$ is a linear function of the Laplacian.

The following proposition is proved in exactly as Theorem 7.

Proposition 15. *1. $ker(\mathcal{I}^*\mathcal{I}) = ker(\mathcal{I})$ is the space of alternating functions on V.*
2. $Range(\mathcal{I}^*\mathcal{I}) = Range(\mathcal{I}^*)$, the orthogonal complement of the alternating functions. When G has no isolated verticies or bipartite components, $Range(\mathcal{I}^*\mathcal{I})$ is all of L_V.
3. $0 \leq \langle f, \mathcal{I}^*\mathcal{I}f\rangle \leq \left(d(G) + 2\sqrt{d_i(G)d_o(G)}\right)\langle f, f\rangle$.

Proposition 16. *1. $\mathcal{I}\mathcal{I}^* = TT^* + SS^* + ST^* + TS^*$*
2. $ker(\mathcal{I}\mathcal{I}^*) = ker(\mathcal{I}^*) = A(G)$, $Range(\mathcal{I}\mathcal{I}^*) = Range(\mathcal{I}) = \Upsilon(G)$.
3. $\mathcal{I}\mathcal{I}^*$ has the same spectrum as $\mathcal{I}^*\mathcal{I}$, and, with the possible exception of the eigenvalue 0, with the same multiplicity. If u is an eigenvector of $\mathcal{I}^*\mathcal{I}$, the $\mathcal{I}u$ is an eigenvector of $\mathcal{I}\mathcal{I}^*$.

Remark: It would be nice to have a characterization of the spectrum in the non-bipartite case. We are unaware of progress in this arena since Grossman et al [9], 1994..

6 The Drift of a Digraph

In this section we discuss a family of fundamental operators on graphs which seems not to have been discussed in the literature. So far we have considered differences and sums of the canonical operators, S, T. At this point we will consider the operator $S + iT$, mapping L_V into L_E. The adjoint operator is $S^* - iT^*$, so the operator $(S^* - iT^*)(S + iT) = S^*S + T^*T + i(S^*T - T^*S)$ is self adjoint and positive semidefinite. We can also consider the same construction using the other square root of -1 to deduce that $(S^* + iT^*)(S - iT) = S^*S + T^*T + i(T^*S - S^*T)$ is self adjoint and positive semidefinite.

Our experience with the difference and incidence operators makes the following lemma a triviality.

Lemma 22. *In a connected digraph G, $S + iT$ (resp. $S - iT$) has a trivial kernel unless the only cycles are of order divisible by 4. In this case there is a one dimensional kernel spanned by complex functions in which the which the magnitude is locally constant and the phase rotates by a factor if i (resp. $-i$) across every edge in traversing from source to target.*

Corollary 3. $(S^* - iT^*)(S + iT)$ *(resp.* $(S^* + iT^*)(S - iT)$*) is invertible unless the only cycles in G have order divisible by 4. In this case there is a one dimensional kernel spanned by complex functions in which the which the magnitude is locally constant and the phase rotates by a factor if i (resp. $-i$) across every edge in traversing from source to target.*

Definition 20. *We will call the operator $\Gamma(G) = T^*S - S^*T$ the* drift operator *of the graph.*

Proposition 17. *The drift operator satisfies the following properties.*

1. *Γ is skew adjoint and real.*
2. *The eigenvalues of Γ are imaginary. For each eigenvector f, $\Gamma f = \lambda f$ implies $\Gamma\overline{f} = -\lambda\overline{f}$, i.e. the eigenvalues and eigenvectors come in complex conjugate pairs.*
3. *Γ generates a one parameter unitary group $t \mapsto e^{t\Gamma}$ on L_V (in fact a group of rotations).*
4. *$\|\Gamma f\| \leq d(G)\|f\|$.*
5. *$\Gamma f(v) = \sum_{e:e^+=v} f(e^-) - \sum_{e:e^-=v} f(e^+)$*

Proof. 1. Since $(S^* + iT^*)(S - iT)$ and $S^*S + T^*T$ are self adjoint, so is $i(S^*T - T^*S)$, hence $S^*T - T^*S$ is skew symmetric. It also maps real functions to real functions.

2. Since $i(S^*T - T^*S)$ is self adjoint, it's eigenvalues are real, hence those of $(S^*T - T^*S)$ are imaginary. Since Γ is real taking the complex conjugate of the eigenvalue equation yields complex conjugate eigenvalues and eigenvectors.
3. This is just the Stone's theorem on generators of unitary groups; since Γ is real, this unitary group is also real, hence a group of rotations.
4. For f a complex function:

$$\begin{aligned} 0 &\leq \langle f, (S^* \pm iT^*)(S \mp iT)f\rangle \\ 0 &\leq \pm i\,\langle f, (S^*T - T^*S)f\rangle + \langle f, S^*Sf\rangle + \langle f, T^*Tf\rangle \\ |\langle f, (S^*T - T^*S)f\rangle| &\leq \langle f, S^*Sf\rangle + \langle f, T^*Tf\rangle \\ |\langle f, (S^*T - T^*S)f\rangle| &\leq d(G)\,\langle f, f\rangle\,. \end{aligned}$$

   But since $(S^*T - T^*S)$ is skew adjoint, its norm is the maximum of the magnitude of its numerical range.
5. This follows from Proposition 2. □

Recall that a *regular graph* is a graph in which the vertex degree $d(v)$ is independent of v and $d(v) = d(G)$. The same condition is less restrictive for digraphs, since $d_i(v)$ and $d_o(G)$ may vary subject to the constraint $d_i(v) + d_o(v) = d(G)$.

Corollary 4. *If the vertex-wise degree $d(v) = d(G)$ is constant then the eigenvectors of Γ are also eigenvectors of $(S^* \mp iT^*)(S \pm iT)$. In particular, if the only cycles in G are of order divisible by 4, then Γ has eigenvalues $\pm d(G)$, and $(S^* \mp iT^*)(S \pm iT)$ has eigenvalue $2d(G)$.*

Note that while the operators Δ and $\mathcal{I}^*\mathcal{I}$ are insensitive to the choice of direction in an edge $e = uv$, changing the sense of an directed edge results in a different Γ by a change of sign in one pair of entries: $\Gamma'_{uv} = -\Gamma_{uv}$. Moreover, in general this change of a direction in a single edge also results in a change of the eigenvalues. However, Γ is insensitive to the addition or deletion of self loops $e = vv$. The following lemma follows from basic properties of rotations.

Lemma 23. $Order(G) \equiv_2 dimker(\Gamma)$. *In particular, when the order of G is odd, Γ has a nonzero kernel.*

Generally the adjacency matrix of a (weighted) graph is given by a symmetric matrix. This data may also be taken as the data of a digraph in which every edge is accompanied by an edge in the opposite direction (of the same weight). We have

Proposition 18. *When the adjacency matrix is symmetric, the drift is zero, i.e. $\Gamma = 0$.*

In this sense the drift operator measures the deviation of a digraph G from a graph.

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