

Chronological algebras: combinatorics and control

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Abstract

This article investigates the geometric and algebraic foundations of exponential product expansions in nonlinear control. A survey of historic developments in geometric control theory on one side, and algebraic combinatorics on the other side exhibits parallel developments and demonstrates how respective findings translate into powerful tools on the other side. Chronological algebras are shown to provide the fundamental structure that unifies geometric, algebraic and combinatorial theories.

1 Introduction

Asymptotic expansions are a fundamental tool for the analysis of nonlinear dynamical systems since closed-form solutions are rarely practical, if at all available. In nonlinear control theory, Lie series expansions, which date back to the work of M. Fliess [11, 12] in the 1970s, have been the most widely used tool for the analysis of optimality and controllability, for realization theory, and for obtaining stabilization results. However, a closer look at the Lie series from a geometric point of view exhibits substantial shortcomings. For example, the series do not immediately exhibit their character as exponential Lie series, and truncations of the series do not correspond to series of any approximating systems.

Considering noncommuting flows as the heart of nonlinear control, it is more natural to work with exponential product expansions. Starting in the late 1970s A. Agrachev and R. Gamkrelidze [1, 2] began to develop a comprehensive theory from this point of view – coining the term *chronological calculus*. While providing the geometric and analytic framework, their work did not yet provide the explicit formulae that allow for efficient calculations in applications. In the mid 1980s H. Sussmann independently managed to derive explicit formulae for infinite product expansions of the Chen-Fliess series [51] by utilizing explicit formulae for Hall bases of free Lie algebras. This work used basic techniques from differential equations (variation of parameters) combined with structural properties of Lie algebras, specifically Lazard elimination. The second section of this article surveys the main strands of this control-theoretic background.

The third section exhibits parallel developments in algebraic combinatorics. Again, the focal points are explicit formulae for infinite exponential product expansions, which in this combinatorial setting have been obtained by Melançon and Reutenauer [37, 38] using rewriting systems (rather than techniques from differential equations). Based on findings by Viennot [55] these explicit formulae are valid for a considerably larger class of bases than the original Hall bases (as defined e.g. in Bourbaki [5]). However, these formulae exhibit a feature that seems to be more than just a cosmetic flaw, as they mix the concatenation and shuffle products, which do belong into different algebras!

The fourth section introduces chronological products as *the* natural product of control theory. It is shown how these products simplify the exponential product expansions, and how they naturally arise when solving differential equations by iteration. As a powerful corollary one obtains explicit formulae (in coordinates) for normal forms of free nilpotent control systems which are widely used as approximating systems, in particular, in path planning algorithms.

The final section provides a more formal description of chronological algebras. Extending recent work by M. Kawski and H. Sussmann [22], abstract chronological algebras are presented as iterated integral functionals. As a highlight the Chen-Fliess series is identified as a resolution of the identity map on an algebra of functions. A few concluding remarks point to related ongoing work in homological algebra which investigates Leibniz algebras which are dual to the chronological algebras that are the foundations of nonlinear control theory.

2 Series and product expansions in control

This work is concerned with finite dimensional control systems on a manifold M

$$\dot{x} = f_0(x) + \sum_{i=1}^m u^i f_i(x). \quad (1)$$

defined by a set of real-analytic vector fields f_i . The controls u^i are measurable and take values in compact intervals, typically containing zero in their interiors. A special case are *systems without drift*, that is systems for which $f_0 \equiv 0$.

Looking for geometric invariants K. T. Chen [6] associated in 1957 to every smooth path $\alpha: [a, b] \mapsto \mathbb{R}^m$ the formal power series

$$\Theta(\alpha) = 1 + \sum_{p=1}^{\infty} \sum_{(i_1, \dots, i_p)} \int_{\alpha} dx_{i_1} \dots dx_{i_p} X_{i_1} \dots X_{i_p} \quad (2)$$

in noncommuting indeterminates X_1, \dots, X_m . The integrals are defined recursively by

$$\int_{\gamma} dx_{i_1} \dots dx_{i_p} = \int_a^b \left(\int_{\gamma^t} dx_{i_1} \dots dx_{i_{p-1}} \right) d\gamma_{i_p}(t) \quad (3)$$

where γ^t denotes the restriction $\gamma|_{[a,t]}$ of the path $\gamma: [a, b] \mapsto \mathbb{R}^m$ to the interval $[a, t]$.

In the 1970 M. Fliess [12, 11] recognized the utility of the Chen series for investigations of nonlinear control systems. Following the notation of Sussmann [49] consider a formal control system

$$\dot{S} = S \left(X_0 + \sum_{i=1}^m u^i X_i \right) \quad (4)$$

initialized at the identity $S(0) = I$ in the associative algebra $\hat{A}(X_0, \dots, X_m)$ of formal power series (with real coefficients) in the noncommuting indeterminates X_i . For controls u^i as above this system has a unique solution. As can be easily verified by direct

substitution, this solution is explicitly given by the *Chen-Fliess series*:

$$\text{Ser}(T, u) = \sum_I \Upsilon^I(T, u) X_I \quad (5)$$

where the sum ranges over all multi-indices $I = (i_1, \dots, i_p)$ with $p \geq 0$ and $1 \leq i_j \leq m$. Here $X_I = X_{i_1} X_{i_2} \dots X_{i_p}$, and

$$\Upsilon^I(T, u) = \int_0^T \int_0^{t_1} \dots \int_0^{t_{p-1}} u^{i_p}(t_p) u^{i_{p-1}}(t_{p-1}) \dots u^{i_1}(t_1) dt_1 dt_2 \dots dt_p \quad (6)$$

For a particular control system (1) initialized at a point $p \in \mathbb{R}^n$, together with an analytic (“*output*”) function $\Phi: \mathbb{R}^m \mapsto \mathbb{R}$, substitute the system vector fields f_j for the indeterminates X_j . Interpreting the products f_I as compositions of first order partial differential operators, one obtains an “*asymptotic series for the propagation of Φ along trajectories of (1)*”

$$\text{Ser}_f(T, u)\Phi = \sum_I \Upsilon^I(T, u)(f_I \Phi)(p) \quad (7)$$

Sussmann [49]] showed that this series converges uniformly, (on sufficiently small time intervals) to Φ along solution curves of the system (1) that are initialized and remain in any given compact subset of M .

Detailed analysis and estimates of the iterated integrals in this series are at heart of the proofs of many sufficient and necessary high-order conditions for small-time local controllability and optimality, compare e.g. [23, 25, 47, 52].

A typical such result by Stefani [47] considers the case of $m = 1$, $|u(\cdot)| \leq 1$, $f_0(p) = 0$. Write $(ad^{k+1}v, w) = [v, (ad^k v, w)]$ to denote iterated Lie brackets of vector fields. Let $\mathcal{S}^1 = \{(ad^k f_0, f_1): k \geq 0\}$, and $\mathcal{S}^{k+1} = \mathcal{S}^k \cup \{[v, w]: v \in \mathcal{S}^1, w \in \mathcal{S}^k\}$. It is shown that if $(ad^{2k} f_1, f_0)(p)$ is linearly independent from the set \mathcal{S}^{2k-1} evaluated at p for some $k \geq 1$ then the system is not small-time controllable about p . This means that for sufficiently small times $T \geq 0$, the initial point p does not lie in the interior of the *reachable set* $\mathcal{R}_p(T)$ (the set of all points that can be reached from p in time T by solution curves of the system (1)).

The proof uses the linear independence of $(ad^{2k} f_1, f_0)(p)$ to find a function Φ such that $\Phi(p) = 0$, $(f_\pi \Phi)(p) = 0$ for all $f_\pi \in \mathcal{S}^{2k-1}$ and $((ad^{2k} f_1, f_0)\Phi)(p) > 0$. After elaborate manipulations of the series (7) Stefani shows that essentially $\Phi(x(T, u)) = ((2k)!)^{-1} \int_0^T (\int_0^s u(s) ds)^{2k} dt + o(|T|^{2k+1}) \geq 0$.

Aside from technical estimates on the size of the iterated integrals, these manipulations involve the rewriting of linear combinations of higher order partial differential

operators as first order operators (Lie-brackets) and combining large numbers of iterated integrals over simplices $\{(t_1, \dots, t_p): 0 \leq t_p \leq \dots \leq t_2 \leq t_1 \leq T\}$ to obtain the structurally much simpler integral $\int_0^T (\int_0^s u(s) ds)^{2k} dt$. For illustration, in the classical case of $k = 1$ (Clebsch-Legendre condition for optimality) typical such steps involve the identities $[f_1[f_1, f_0]] = f_1 f_1 f_0 - 2f_1 f_0 f_1 + f_0 f_1 f_1$ and $\int_0^T (\int_0^t u^1(s))^2 u^0(t) dt = 2 \int_0^T \int_0^{t_1} \int_0^{t_2} u^0(t_1) u^1(t_2) u^1(t_3) dt_3 dt_2 dt_1$.

The book-keeping issues become horrendous, but it also becomes clear that there are some hidden algebraic and combinatorial structures that make these manipulations possible. These algebraic structures were partially utilized for a partial factorization of the Chen-Fliess series (5) (factorization of the infinite partial sum corresponding to terms involving at most three factors f_1) in the proof [23] of a necessary condition involving the iterated Lie bracket $[[f_0, f_1], [[f_0, f_1], f_0]]$. A complete factorization of the Chen-Fliess series (5) was achieved by Sussmann [51], but it necessitated the additional structural properties of Lie algebras and Hall sets, see below.

As successful as the Chen-Fliess series was for deriving the early conditions for controllability and optimality, it exhibits some serious drawbacks. First of all the number of terms is very large – more specifically, the number of terms of total degree less or equal to N is of the order of $(m + 1)^{N-1}$. For practical purposes it is cumbersome that truncations of the Chen-Fliess series, which one might want to use for approximations of solutions, do not correspond to series expansions for any approximate systems. Closely related is that intricate cancellations among the terms of the Chen-Fliess series make it virtually impossible to decide accessibility, i.e. to decide whether all solution curves of a system (starting from a fixed initial point) lie within a nontrivial submanifold of the state-space. (The standard infinitesimal criterion for accessibility is the Lie algebra rank condition: The system (1) is accessible from p if and only if the set of all iterated Lie brackets of the vector fields f_0, \dots, f_m spans the tangent space $T_p M$.) But most importantly, it is clear from elementary considerations that the Chen-Fliess series (5) is an exponential Lie series, i.e. there exists a Lie series $\Xi(T, u)$ (whose coefficients depend on the controls) such that $\text{Ser}(T, u) = \exp(\Xi(T, u))$. This may be seen using differential equations arguments related to the Campbell-Baker-Hausdorff formula [48, 51], or via the combinatorial argument (Ree's theorem [42, 43]) that the iterated integrals Υ^I of the Chen-Fliess series (5) satisfy the *shuffle relations*

$$\Upsilon^I \cdot \Upsilon^J = \Upsilon^{I \sqcup J} \tag{8}$$

for all multi-indices I and J . (The next section provides a detailed discussion of the shuffle product and its algebra.)

For practical purposes one may consider any ordered basis $\mathcal{B} = \{B_1, B_2, \dots\}$ for the free Lie algebra $L(X_0, \dots, X_m)$, and ask for explicit formulae for the coefficients $\alpha^k(T, u)$ in the series $\log(\text{Ser}(T, u)) = \Xi(T, u) = \sum_{k=1}^{\infty} \alpha^k(T, u) B_k$ as functionals on the space of controls u . (Unless otherwise specified all algebras are assumed to have coefficients in an infinite field \mathbf{k} of characteristic zero.) A general formula for such *coordinates of the first kind* has been provided in [51] in terms of a spanning set of the free Lie algebra $L(X_0, \dots, X_m)$. This provides a significant reduction in the number of terms (as compared to the original Chen-Fliess series (5) as the dimension of subspace of the free Lie algebra on $(m + 1)$ generators that is spanned by all iterated products of the generators of length less than N is equal to $\frac{1}{N} \sum_{k|N} \mu(k) (m + 1)^{N/k}$ (compare e.g. [5] §3.3) where μ is the Moebius function ($\mu(n) = 0$ if there exists a prime p such that $p^2 | n$, and $\mu(n) = (-1)^s$ if n is the product of s distinct primes). Alternatively, one may ask for explicit formulae for *coordinates of the second kind*, i.e. explicit formulae for the coefficients $\beta^B(T, u)$ in an infinite directed product expansion

$$\text{Ser}(T, u) = \prod_{B \in \mathcal{B}}^{\leftarrow \infty} \exp(\beta^B(T, u) B) \quad (9)$$

where $\mathcal{B} \subseteq L_{\mathbf{k}}(\mathcal{X})$ is any suitable basis of the free Lie algebra. This is the route successfully taken by Sussmann [51], and also by Melançon and Reutenauer [37, 38] in the combinatorial setting. This product expansion relies on the basic differential equations technique of *variation of parameters* combined with structural properties of Hall sets and Lazard elimination [5]: Suppose \mathbf{k} a field of scalars, \mathcal{X} is a set and $c \in \mathcal{X}$. Then the free Lie algebra $L_{\mathbf{k}}(\mathcal{X})$ over \mathbf{k} generated by \mathcal{X} is the direct sum of the one-dimensional subspace $\{\lambda c : \lambda \in \mathbf{k}\}$ and of a Lie-subalgebra of $L_{\mathbf{k}}(\mathcal{X})$ that is freely generated by the set $\{(\text{ad}^j c, b) : b \in \mathcal{X} \setminus \{c\}, j \geq 0\}$.

The origins of the first bases for free Lie algebras go back to the work of Ph. Hall [17] who investigated commutator groups, and they were formalized in a Lie algebra setting by M. Hall [15, 16]. Several other bases were proposed in subsequent years, most notably *Lyndon bases* [7] and *Siršov bases* [46], which initially were regarded as structurally different. However, Viennot [55] showed that all these bases arose from the same fundamental principles (31). In the following we will refer to these as *generalized Hall bases*, or as *Hall-Viennot bases*. To characterize these bases start with a set \mathcal{X} . The free magma $\mathcal{M}(\mathcal{X})$ over \mathcal{X} is the set of all *parenthesized* words, i.e. the union $\mathcal{M}(\mathcal{X}) = \cup_{n=1}^{\infty} \mathcal{M}^n(\mathcal{X})$ where the sets $\mathcal{M}^n(\mathcal{X})$ are defined inductively by $\mathcal{M}^1(\mathcal{X}) = \mathcal{X}$ and $\mathcal{M}^n(\mathcal{X}) = \cup_{i+j=n} \mathcal{M}^i(\mathcal{X}) \times \mathcal{M}^j(\mathcal{X})$.

A generalized Hall set over a set \mathcal{X} is any strictly ordered subset $\tilde{\mathcal{H}} \subseteq \mathcal{M}(\mathcal{X})$ that

satisfies [55] (in the literature one frequently finds definitions which reverse the ordering $<$, and/or which consider the words read from right to left)

- $\mathcal{X} \subseteq \tilde{\mathcal{H}}$
- Suppose $a \in \mathcal{X}$. Then $(w, a) \in \tilde{\mathcal{H}}$ iff $w \in \mathcal{H}$, $w < a$ and $a < (w, a)$. (10)
- Suppose $u, v, w, (u, v) \in \tilde{\mathcal{H}}$.
Then $(u, (v, w)) \in \tilde{\mathcal{H}}$ iff $v \leq u \leq (v, w)$ and $u < (u, (v, w))$.

The original Hall elements (in a narrow sense) as defined e.g. in Bourbaki [5] require that the ordering of the Hall elements be compatible with the length of the word, i.e. $u, v \in \tilde{\mathcal{H}}$ and $u < v$ imply that $|u| \leq |v|$. Viennot showed that one can omit this condition, and still obtain bases for free Lie algebras.

The importance of Hall sets for our purposes arises from the fact that their images under the canonical map $\varphi: \mathcal{M}(\mathcal{X}) \mapsto L_{\mathbf{k}}(\mathcal{X})$ from the free magma into the free Lie algebra are bases for $L_{\mathbf{k}}(\mathcal{X})$ [5, 16, 55]. Here,

$$\varphi(a) = a \text{ for } a \in \mathcal{X}, \text{ and } \varphi((w, z)) = [\varphi(w), \varphi(z)] \text{ for } w, z \in \mathcal{M}(\mathcal{X}). \quad (11)$$

The restriction of the map φ to any generalized Hall-set $\tilde{\mathcal{H}} \subseteq \mathcal{M}(\mathcal{X})$ is one-to-one by construction [55]. Hence the inverse image $\varphi^{-1}(H) \in \tilde{\mathcal{H}} \subseteq \mathcal{M}(\mathcal{X})$ of an element $H \in \varphi(\tilde{\mathcal{H}}) \subseteq L_{\mathbf{k}}(\mathcal{X})$ is well-defined, and we denote it by \tilde{H} for notational convenience. Caution is advised with this notation as, for example, one may have $\{x, y, (x, y), (y, (x, (x, y)))\} \subseteq \tilde{\mathcal{H}} \subseteq \mathcal{M}(\{x, y\})$. Then $\varphi((x, y)) = [x, y] = -[y, x]$, and $\varphi((y, (x, (x, y)))) = [y, [x, [x, y]]] = [x, [y, [x, y]]]$ (due to the anti-commutativity and Jacobi-identity in $L_{\mathbf{k}}(\mathcal{X})$). Consequently, $\varphi^{-1}([x, [y, [x, y]]]) = \varphi^{-1}([y, [x, [x, y]]]) = (y, (x, (x, y))) \neq (x, (y, (x, y)))$ in $\mathcal{M}\mathcal{X}$ (unless $x = y$). This distinction between formal brackets, or parenthesized words in $\mathcal{M}(\mathcal{X})$ on one side, and elements of the free Lie algebra $L_{\mathbf{k}}(\mathcal{X})$ on the other side is necessary as Hall-sets, and the iterated integrals depend critically on the factorization of the Hall words. However, since $[x, y] = [-y, x]$ in any Lie algebra it is impossible to define left and right factors of Lie-brackets.

Sussmann's product expansion [51] of the Chen-Fliess series (5) originally used Hall-sets in the narrower sense as found Bourbaki [5]. But it can be shown that with some minor modifications the original proof also holds for the more general Hall-Viennot bases. Specifically, consider a generalized Hall set $\tilde{\mathcal{H}} = \{\tilde{H}_1, \tilde{H}_2, \dots\} \subseteq \mathcal{M}(\mathcal{X})$ (with $\mathcal{X} = \{0, 1, \dots, m\}$). Define two sets of iterated integrals that are related via $C^{\tilde{H}}(T, u) = \int_0^T c^{\tilde{H}}(t, u) dt$ for $\tilde{H} \in \tilde{\mathcal{H}}$, and defined recursively by $c^a(t, u) = u^a(t)$ for $a \in \mathcal{X}$ and

$$c^{\tilde{H}}(t, u) = \frac{1}{m!} \left(C^{\tilde{H}_1}(t, u) \right)^m \cdot c^{\tilde{H}_2}(t, u) \text{ if } \tilde{H} = \underbrace{(\tilde{H}_1, (\tilde{H}_1, \dots, (\tilde{H}_1, \tilde{H}_2)) \dots)}_{m\text{-times}} \in \tilde{\mathcal{H}} \quad (12)$$

and either $\tilde{H}_2 \in \mathcal{X}$ or $\tilde{H}_2 = (\tilde{H}_{21}, \tilde{H}_{22})$ with $\tilde{H}_1 \neq \tilde{H}_{21}$ and $\tilde{H}_{21}, \tilde{H}_{22} \in \tilde{\mathcal{H}}$. The main theorem of [51] states that these iterated integrals are the coefficients of the exponential product expansion (9) of the Chen-Fliess-series – very carefully written as $C^{\tilde{H}}(T, u) = \beta^{\varphi(\tilde{H})}(T, u)$ for $\tilde{H} \in \tilde{\mathcal{H}} \subseteq \mathcal{M}(\mathcal{X})$.

It is informative to revisit some key-steps of Sussmann’s original proof [51] as they reveal very close links to key structural properties of both chronological algebras and Hall-sets, linking combinatorics and differential equations. For the sake of notational convenience consider the system without drift

$$\dot{S} = S \left(\sum_{i=1}^m u^i X_i \right) \quad (13)$$

initialized at the identity $S(0) = I$ in the associative algebra $\hat{A}_{\mathbf{k}}(X_1, \dots, X_m)$ of formal power series in the noncommuting indeterminates X_i (with coefficients in a field \mathbf{k} of characteristic zero). (The system (4) is obtained via renumbering and setting $u^0 \equiv 1$.) Choose an ordered Hall set $\tilde{\mathcal{H}}$. For any formal bracket $\tilde{H} \in \tilde{\mathcal{H}}$ denote the image $\varphi(\tilde{H})$ in the free Lie algebra $L_{\mathbf{k}}(X_1, \dots, X_m)$ by H . Iteratively using the technique of variation of parameters the solution may be expressed in the form

$$S(t) = S_j(t) \prod_{H \in \mathcal{F}_j}^{\leftarrow} e^{\beta^H(t, u)H} \quad (14)$$

where $\mathcal{F}_j \subseteq \mathcal{H}$ is an increasing family of subsets constructed iteratively, starting with $\mathcal{F}_0 = \emptyset$. At each stage the varied parameter S_j , starting with $S_0 = S$, satisfies its own differential equation

$$\dot{S}_j(t) = S_j(t) \left(\sum_{H \in \mathcal{G}_j} \dot{\beta}^H(t, u)H \right) \quad (15)$$

where the subsets $\mathcal{G}_j \subseteq \mathcal{H}$ are constructed iteratively, starting with $\mathcal{G}_0 = \{X_1, \dots, X_m\}$ the original set of generators. At each stage $j \geq 1$ select the least element $H_j^* \in \mathcal{G}_{j-1}$ (with respect to the ordering induced from $\tilde{\mathcal{H}}$), and define $\mathcal{F}_j = \mathcal{F}_{j-1} \cup \{H_j^*\}$. By differentiating (14) (using the new value for j) and substituting into (13) one obtains, after some basic manipulations [51], the new differential equation (15) where the next set \mathcal{G}_j must be chosen as

$$\mathcal{G}_j = \{(ad^\nu H_j^*, H) : H \in \mathcal{G}_{j-1}, H \neq H_j^*\} \quad (16)$$

Note that by choice of $H_j^* \in \mathcal{G}_{j-1}$ all elements of \mathcal{G}_j are again Hall-elements, i.e. $\mathcal{G}_j \in \mathcal{H}$. Using Lazard’s elimination theorem one observes that the free Lie algebra

$L(\{X_1, \dots, X_m\})$ is the direct sum of the one-dimensional subspaces spanned by the elements of \mathcal{F}_j and a subalgebra generated freely by the new set \mathcal{G}_j .

The key step to obtain a formula for the iterated integral coefficients $\beta^H(t, u)$ repeatedly uses the basic fact that for any two elements X , and Y of a Lie algebra

$$e^{tX} Y e^{-tX} = \sum_{\nu \geq 0} \frac{1}{\nu!} t^\nu (\text{ad}^\nu X, Y). \quad (17)$$

More specifically, the technical manipulations leading to the differential equation (15) for the next value of j involve the critical step:

$$e^{\beta_j^*(t, u)} \left(\sum_{H \in \mathcal{G}_{j-1} \setminus \{H_j^*\}} \left(\frac{d}{dt} \beta^H(t, u) \right) H \right) e^{-\beta_j^*(t, u)} = \sum_{\nu \geq 0} \sum_{H \in \mathcal{G}_{j-1} \setminus \{H_j^*\}} \frac{1}{\nu!} \left(\beta^{H_j^*}(t, u) \right)^\nu \cdot \left(\frac{d}{dt} \beta^H(t, u) \right) \cdot (\text{ad}^\nu H_j^*, H) \quad (18)$$

From the coefficients of the iterated brackets $(\text{ad}^\nu H_j^*, H)$ in this expression one directly obtains the formula (12) for the iterated integrals $\beta^{(\text{ad}^\nu H_j^*, H)}(t, u)$. For technical details, and precise discussion of convergence etc. see the original reference [51].

This exponential product expansion has found numerous applications. The proofs of the known high-order sufficient and necessary conditions simplify considerably as now much fewer iterated integrals need to be analyzed, and these iterated integrals have very special structures, allowing one to more quickly find upper and lower bounds. For an efficient calculation related to the classical Clebsch-Legendre condition see [22]. In a similar style the necessary conditions of Stefani [47] and Kawski [23] may also be derived much more efficiently [26].

The most successful applications of the product expansion may lie in many path planning algorithms, e.g. [18, 19, 27, 40, 41, 53]. These are closely related to normal forms for free nilpotent control systems, which may be immediately read off from the explicit formula for the iterated integrals β^H . With a slight abuse of notation use de-parenthesized Hall words (compare the next section) to index the coordinates x_H and consider the control system

$$\begin{aligned} \dot{x}_a &= u_a && \text{if } a \in \mathcal{X} \\ \dot{x}_{HK} &= x_H \dot{x}_K && \text{if } H, K, HK \in \mathcal{H} \subseteq \text{cal}W(\mathcal{X}) \end{aligned} \quad (19)$$

The suggestive notation employed here is to imply that H and K are the uniquely determined left and right factors of the Hall word HK (compare the next section for

details). For any control $U(t)$ one directly reads off the value of the coordinate functions along the solution curve (starting at $x = 0$

$$x_H(t) = m_H \cdot \beta^H(t, u) \tag{20}$$

where m_H is a multi-factorial as apparent from the recursive formulas for $\beta^H(t, u)$. Essentially the same normal forms for free nilpotent systems have been obtained by Grayson and Grossman [14] who verified, essentially by direct calculation, that the vector fields defining the system (19) generate a free Lie algebra. Independently of these developments, Agrachev and Gamkrelidze [1, 2] developed a completely geometric formalism for dealing with compositions of flows of time-varying vector fields. This *chronological calculus* avoided the pitfalls of the Chen Fliess series, but it lacked the explicit formulae of Sussmann’s exponential product expansion that were made possible by utilizing specific bases for free Lie algebras. The later sections of this article will revisit the chronological formalism in detail and further develop the algebraic background.

3 Combinatorics of exponential products

The applications of the product expansion to control make it natural to use differential equations techniques, as in the preceding section. However, it also is clear that underlying the manipulations of the iterated integrals there are some basic algebraic and combinatorial structures. The combinatorial objects contain just the essential information needed for the manipulations and avoid all the excessive notational burden of the iterated integrals. The goal of this section is to exhibit the (combinatorial) analogue of the exponential product expansion of the Chen-Fliess series as a factorization of the identity map on the space of noncommuting formal power series. To this end we review the properties of some basic algebraic and combinatorial objects. Throughout we will largely follow the terminology of Lothaire [34].

The set of indeterminates $\mathcal{X} = \{X_1, \dots, X_m\}$ is commonly called an *alphabet*. (In general the alphabet may be infinite, but for our purposes it suffices to consider finite alphabets.) We will use the variables a, b, c, \dots to denote its elements, which are also referred to as *letters*. The free monoid over \mathcal{X} , denoted $W(\mathcal{X})$ is the set of all *words* (or noncommuting monomials) endowed with the noncommutative, associative *concatenation product* $(w, z) \mapsto wz$. The empty word is denoted by 1 and satisfies $1w = w1 = w$ for all $w \in W(\mathcal{X})$. Write $W_0(\mathcal{X}) = W(\mathcal{X}) \setminus \{1\}$ for the set of nonempty

words. Throughout, \mathbf{k} denotes a field of scalars of characteristic zero. The associative algebras of noncommutative polynomials noncommutative formal power series (with coefficients in \mathbf{k}) are denoted by $A_{\mathbf{k}}(\mathcal{X})$ and $\hat{A}_{\mathbf{k}}(\mathcal{X})$, respectively. Associated with the basis $W(\mathcal{X})$ is the pairing $\langle \cdot, \cdot \rangle : \hat{A}_{\mathbf{k}}(\mathcal{X}) \times A_{\mathbf{k}}(\mathcal{X}) \mapsto \mathbf{k}$ defined by

$$\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \sum_{w \in W(\mathcal{X})} \alpha_w \beta_w \quad \text{for } \alpha = \sum_{w \in W(\mathcal{X})} \alpha_w w \in \hat{A}_{\mathbf{k}}(\mathcal{X}), \beta = \sum_{w \in W(\mathcal{X})} \beta_w w \in A_{\mathbf{k}}(\mathcal{X}) \quad (21)$$

With this pairing $\hat{A}_{\mathbf{k}}(\mathcal{X})$ is the algebraic dual of $A_{\mathbf{k}}(\mathcal{X})$, and $A_{\mathbf{k}}(\mathcal{X})$ is the topological dual of $\hat{A}_{\mathbf{k}}(\mathcal{X})$ with the usual topologies, compare [22] for details. Use $A_{\mathbf{k},0}(\mathcal{X})$ and $\hat{A}_{\mathbf{k},0}(\mathcal{X})$ to denote the sets of noncommutative polynomials and power series with zero constant term, respectively, i.e.

$$\begin{aligned} A_{\mathbf{k},0}(\mathcal{X}) &= \{ \alpha \in A_{\mathbf{k}}(\mathcal{X}) : \langle 1, \alpha \rangle = 0 \} \\ \hat{A}_{\mathbf{k},0}(\mathcal{X}) &= \{ \alpha \in \hat{A}_{\mathbf{k}}(\mathcal{X}) : \langle \alpha, 1 \rangle = 0 \} \end{aligned} \quad (22)$$

The algebra $\hat{A}_{\mathbf{k}}(\mathcal{X})$ is equipped with a natural co-product: the diagonal map

$$\Delta: \hat{A}_{\mathbf{k}}(\mathcal{X}) \times \hat{A}_{\mathbf{k}}(\mathcal{X}) \mapsto \hat{A}_{\mathbf{k}}(\mathcal{X}) \quad (23)$$

is the \mathbf{k} -algebra homomorphism that is defined on generators $a \in \mathcal{X}$ by $\Delta(a) = a \otimes 1 + 1 \otimes a$. Its transpose is the bilinear map $\mathbb{W}: A_{\mathbf{k}}(\mathcal{X}) \times A_{\mathbf{k}}(\mathcal{X}) \mapsto A_{\mathbf{k}}(\mathcal{X})$ defined by

$$\langle u, w \mathbb{W} z \rangle = \langle \Delta(u), w \otimes z \rangle \quad \text{for all } u \in \hat{A}_{\mathbf{k}}(\mathcal{X}), \quad w, z \in A_{\mathbf{k}}(\mathcal{X}) \quad (24)$$

For a detailed discussion of the Hopf-algebra structures compare Reutenauer [44, 54]. Alternatively, one may characterize the shuffle product combinatorially by defining $1 \mathbb{W} w = w \mathbb{W} 1 = w$ for all $w \in W(\mathcal{X})$ and, for $a, b \in \mathcal{X}$ and $w, z \in W(\mathcal{X})$,

$$(wa) \mathbb{W} (zb) = ((wa) \mathbb{W} z) b + (w \mathbb{W} (zb)) a. \quad (25)$$

This recursive characterization is useful to clarify how the shuffle product is related to the iterated integrals. Using Ree's theorem [42, 43, 54] one may establish that the Chen-Fliess series is an exponential Lie series (compare the previous section) by showing that the coefficients $\Upsilon^I(T, u)$ satisfy the shuffle relations (8).

More precisely, on a suitable space \mathcal{U} of controls, let $\mathcal{IIF}(\mathcal{X})$ be the algebra of *iterated integral functionals* considered as a subalgebra of all mappings from $\mathcal{U}^{\mathcal{X}}$ into \mathcal{U} (compare the next sections, or [22], for technical details). Using the terminology introduced

in this section the linear map $\Upsilon: A(\mathcal{X}) \mapsto \mathcal{IIF}(\mathcal{X})$ of (6) may be defined recursively by

$$\begin{aligned} \Upsilon(a) &: (T, u) \mapsto \int_0^T u^a(t) dt && \text{for } a \in \mathcal{X} \\ \Upsilon(wa) &: (T, u) \mapsto \int_0^T \Upsilon^w(t, u) u^a(t) dt && \text{for } a \in \mathcal{X}, 1 \neq w \in W(\mathcal{X}) \end{aligned} \quad (26)$$

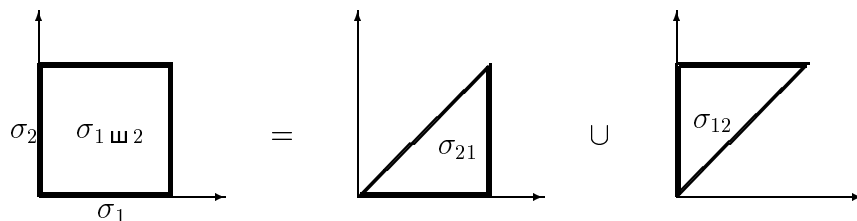
We use the symbols $\Upsilon(wa)$ and Υ^{wa} interchangeably. To *satisfy the shuffle relation* means that the map Υ is an associative algebra homomorphism from the algebra $A(\mathcal{X})$ equipped with the shuffle product to the algebra of iterated integral functionals under pointwise multiplication.

It is instructive to take a closer look at a direct proof by induction on the combined length of the words (wa) and (zb) . The empty word 1 is mapped to the iterated integral functional $\Upsilon^1(t, u) \equiv 1$. For any letter $a \in \mathcal{X}$ it is clear that $\Upsilon^a \sqcup^1(t, u) = \Upsilon^a(t, u) = \Upsilon^a(t, u) \cdot 1 = \Upsilon^a(t, u) \cdot \Upsilon^1(t, u)$. The induction step uses the recursive definitions of the shuffle product (25) and of the map Υ . The calculation shows clearly how shuffle product of the indices corresponds to the pointwise multiplication of the iterated integrals, and how the recursive characterization (25) of the shuffle product encodes integration by parts.

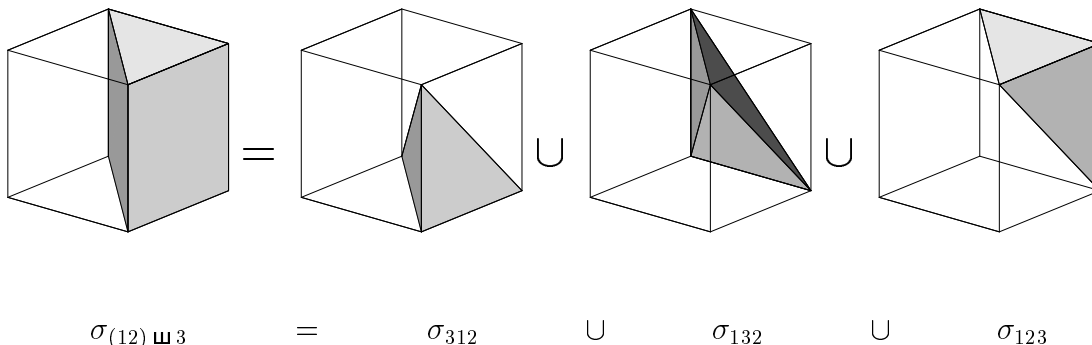
$$\begin{aligned} \Upsilon^{(wa) \sqcup (zb)}(u, T) &= \\ &= \Upsilon^{((wa) \sqcup z)^b + (w \sqcup (zb))^a}(u, T) && \text{(using (25))} \\ &= \Upsilon^{((wa) \sqcup z)^b}(u, T) + \Upsilon^{(w \sqcup (zb))^a}(u, T) && \text{(linearity)} \\ &= \int_0^T \Upsilon^{(wa) \sqcup z}(t, u) \cdot u^b(t) dt + \int_0^T \Upsilon^{w \sqcup (zb)}(t, u) \cdot u^a(t) dt && \text{(using (26))} \\ &= \int_0^T \left(\Upsilon^{wa}(t, u) \cdot \Upsilon^z(t, u) \cdot u^b(t) + \Upsilon^w(t, u) \cdot \Upsilon^{zb}(t, u) \cdot u^a(t) \right) dt \\ &&& \text{(by induction hypothesis)} \\ &= \int_0^T \left(\Upsilon^{wa}(t, u) \cdot \frac{d}{dt} \Upsilon^{zb}(t, u) + \frac{d}{dt} (\Upsilon^{wa}(t, u)) \cdot \Upsilon^{zb}(t, u) \right) dt \\ &&& \text{(using (26))} \\ &= \Upsilon^{wa}(T, u) \cdot \Upsilon^{zb}(T, u) && \text{(integration by parts)} \end{aligned} \quad (27)$$

In the literature, one also finds *partially commutative* shuffle products [10], and a noncommutative shuffle product [3] on spaces of permutations – this latter is closely related to the product expansion and the iterated integrals. The following illustrations serve to suggestively demonstrate how permutations may be mapped to the iterated integrals over simplices as they occur in the Chen-Fliess series, and how the shuffle product of permutations corresponds to the Cartesian product of simplices, which in turn may again be decomposed into sums of simplices.

For a word $w = (i_1, \dots, i_m)$ containing each of the letters $1, 2, \dots, n$ at most once define $\sigma_w = \{(t_1, \dots, t_n) : 0 \leq t_{i_1} \leq t_{i_2} \leq \dots \leq t_{i_m} \leq 1, \text{ and } t_k = 0 \text{ if } i_j \neq k \text{ for } 1 \leq j \leq m\}$, and extend σ linearly to the group algebra. For example, if $n = 2$ then $\sigma(1) = [0, 1] \times \{0\}$, $\sigma(2) = \{0\} \times [0, 1]$, $\sigma(21) = \{(t_1, t_2) : 0 \leq t_2 \leq t_1 \leq 1\}$, and the shuffle product $(1 \sqcup 2) = 12 + 21$ maps to the Cartesian product $\sigma_{1 \sqcup 2} = [0, 1] \times [0, 1]$.



The following is typical example in the case of three letters $\{1, 2, 3\}$



$$0 \leq t_1 \leq t_2 \leq 1, \quad 0 \leq t_3 \leq 1$$

For multiplicative integrands $f(x, y, z) = f_1(x) \cdot f_2(y) \cdot f_3(z)$ this example leads to the following suggestive formula (using x, y, z , instead of t_1, t_2, t_3 for better readability):

$$\left(\int_0^1 \int_0^y (\cdot) dx dy \right) \cdot \int_0^1 (\cdot) dz = \int_0^1 \int_0^y \int_0^x (\cdot) dz dx dy + \int_0^1 \int_0^y \int_0^z (\cdot) dx dz dy + \int_0^1 \int_0^z \int_0^y (\cdot) dx dy dz$$

The partial factorization of the Chen-Fliess series as in [23] made repeated use of such identities (going from the right to the left side).

Thus the composition and products of the iterated integral functionals are combinatorially encoded using the shuffle product of the algebra $A_{\mathbf{k}}(\mathcal{X})$. The transpose of the shuffle product is the diagonal map on the algebra $\hat{A}_{\mathbf{k}}(\mathcal{X})$ (23). Inside this algebra $\hat{A}_{\mathbf{k}}(\mathcal{X})$ of formal power series the noncommutative polynomials (or finite linear combinations of words) correspond to partial differential operators (of finite order) that are obtained as compositions of the vector fields defining the original control

system (1). Among the infinite linear combinations in $\hat{A}_{\mathbf{k}}(\mathcal{X})$ there are also some distinguished elements which may be regarded as points (or translations, or evaluation at points) on a formal group. Here we review the combinatorial properties of these distinguished elements of $\hat{A}_k(\mathcal{X})$. Define *Lie polynomials* to be the elements of the smallest subspace of $\hat{A}(\mathcal{X})$ that contains \mathcal{X} and that is closed under the Lie bracket $[\cdot, \cdot]: (w, z) \mapsto [w, z] = wz - zw$. Note that this subspace may be identified with the free Lie algebra $L(\mathcal{X})$ over \mathcal{X} . Furthermore denote by $\hat{L}(\mathcal{X})$ the set of *Lie series*, that is the set of all elements $s = \sum_{n=1}^{\infty} s_n \in \hat{A}(\mathcal{X})$ all of whose homogeneous components $s_n = \sum_{|w|=n} \langle s, w \rangle w \in L(\mathcal{X})$ are Lie polynomials. (Here $|w|$ denotes the length of the word $w \in W(\mathcal{X})$.)

For letters $a, b \in \mathcal{X}$, one easily sees that $\Delta(ab) = (ab) \otimes 1 + a \otimes b + b \otimes a + 1 \otimes (ab)$, and hence $\Delta([a, b]) = \Delta(ab) - \Delta(ba) = ((ab) \otimes 1 + 1 \otimes (ab)) - ((ba) \otimes 1 + 1 \otimes (ba)) = [a, b] \otimes 1 + [b, a] \otimes 1$. Repeating this argument inductively, one obtains the *Friederich's criterion* that an element $s \in \hat{A}(\mathcal{X})$ is a Lie series if and only if $\Delta(s) = s \otimes 1 + 1 \otimes s$. Using the definition of the shuffle as the transpose of the diagonal map one obtains as a direct consequence that the $\tilde{L}(\mathcal{X})$ is orthogonal to all nontrivial shuffles. Suppose $\ell \in L(\mathcal{X})$. Then

$$\begin{aligned} \langle \ell, w \sqcup z \rangle &= \langle \Delta(\ell), w \otimes z \rangle \\ &= \langle \ell \otimes 1, w \otimes z \rangle + \langle 1 \otimes \ell, w \otimes z \rangle \\ &\neq 0 \quad \text{only if } w = 1 \text{ or } z = 1. \end{aligned} \tag{28}$$

From this criterion for Lie series one easily obtains a similar criterion for a series $\Phi \in \hat{A}(\mathcal{X})$ to be an exponential Lie series [44], i.e. that there exists a Lie series $\Xi \in \hat{L}(\mathcal{X})$ (denoted $\log(\Phi)$) such that $\Phi = \exp(\Xi) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \Xi^n$.

$$\Delta(\Phi) = \Phi \otimes \Phi \iff \log(\Phi) \in \hat{L}(\mathcal{X}) \tag{29}$$

In terms of the shuffle product this immediately translates into Ree's theorem [42, 43] that a series $\Phi \in \hat{A}(\mathcal{X})$ is an exponential Lie series if and only if its coefficients $\langle \Phi, w \rangle$, $w \in W(\mathcal{X})$, satisfy the *shuffle relations*: Indeed, since $\langle \Phi, w \sqcup z \rangle = \langle \Delta(\Phi), w \otimes z \rangle$ the condition $\Delta(\Phi) = \Phi \otimes \Phi$ is equivalent to

$$\langle \Phi, w \sqcup z \rangle = \langle \Phi, w \rangle \cdot \langle \Phi, z \rangle \tag{30}$$

This property justifies the name *group-like* elements for the exponential Lie series.

Before we can return to the exponential product expansion we need to revisit Hall bases. Bases for free Lie algebras proposed by M. Hall [15, 16], Lyndon [7], and Siršov [46]

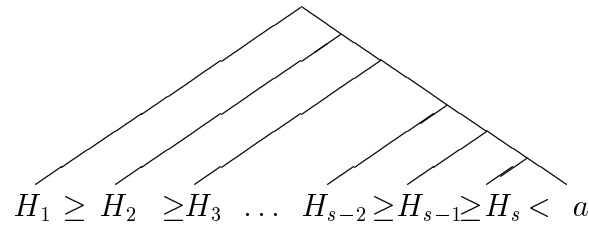
were originally considered to be distinct, until Viennot [55] showed that they all arise from a fundamental *unique factorization principle*, leading to the definition we gave above (10). A consequence of the unique factorization principle is that the restriction of the canonical map $\psi: \mathcal{M}(\mathcal{X}) \mapsto W(\mathcal{X})$ (from the free magma into the free monoid, $\psi(a) = a$ for $a \in \mathcal{X}$, and $\psi(\tilde{w}, \tilde{z}) = \psi(\tilde{w})\psi(\tilde{z})$ for $\tilde{w}, \tilde{z} \in \mathcal{M}(\mathcal{X})$) to any generalized Hall-set $\tilde{\mathcal{H}} \subseteq \mathcal{M}(\mathcal{X})$ is one-to-one. This effectively allows us to conveniently “*drop the parentheses*”, identifying unparenthesized Hall words H with parenthesized elements \tilde{H} of the Hall-set. The unique factorization principle [55] that underlies the construction of these bases asserts that for any such fixed generalized Hall-set $\tilde{\mathcal{H}}$, now considered as a subset of $W(\mathcal{X})$, every word $w \in W(\mathcal{X})$ has a unique factorization into a nonincreasing product of Hall words, i.e. there exist unique $H_j \in \mathcal{H}$, such that

$$w = H_1 H_2 \dots H_s \quad \text{and} \quad H_1 \geq H_2 \geq \dots \geq H_s \quad (31)$$

In particular, this allows one to quickly insert brackets or parenthesis into a Hall-word to find the images under the inverse map $\psi^{-1}: \psi(\tilde{\mathcal{H}}) \mapsto \mathcal{M}(\mathcal{X})$: Any Hall-word $H \in \psi(\tilde{\mathcal{H}} \setminus \mathcal{X}) \subseteq W(\mathcal{X}) \setminus \mathcal{X}$, that is not a generator, may be written in the form $H = Ka$ with $a \in \mathcal{X}$ and $w \in W(\mathcal{X})$. Then w uniquely factorizes as above and the parenthesation map ψ^{-1} gives

$$\psi^{-1}(H) = \psi^{-1}(H_1 H_2 \dots H_s a) = \left(\psi^{-1}(H_1) \left(\psi^{-1}(H_2) \left(\dots \left(\psi^{-1}(H_s) a \right) \dots \right) \right) \right) \quad (32)$$

This special structure is nicely illustrated pictorially by the structure of the binary tree:



In complete analogy, the bracketing is the linear map $[\cdot]: \psi(A_{\mathbf{k}}(\mathcal{X})) \mapsto A_{\mathbf{k}}(\mathcal{X})$ defined for generators $a, b \in \mathcal{X}$ with $a < b$ by $[a] = a$ and $[a, b] = ab - ba$, and recursively on words $w \in W(\mathcal{X})$ with unique factorization $w = H_1 H_2 \dots H_s$ by

$$[H_1 H_2 \dots H_{s-1} H_s] = [[H_1], [[H_2], [\dots, [[H_{s-1}], [H_s]]] \dots]] \quad (33)$$

Note that for $a < b$ as above this implies $[b, a] = ba$ (whereas $[a, b] = ab - ba$). For a Hall set $\tilde{\mathcal{H}} \subseteq \mathcal{M}(\mathcal{X})$ the map $[\cdot]$ maps the image $\psi(\tilde{\mathcal{H}}) \subset W(\mathcal{X})$ to a subset of the free Lie algebra, in particular, to a basis of $L_{\mathbf{k}}(\mathcal{X})$. In terms of the map φ , compare (11), one may consider $[\cdot]$ an extension of $\varphi \circ \psi^{-1}$ from $\psi(\tilde{\mathcal{H}})$ to $A_{\mathbf{k}}\mathcal{X}$.

For any ordered basis \mathcal{B} of the free Lie algebra $L_{\mathbf{k}}(\mathcal{X})$, the set $\mathcal{P} = \{B_{i_1}B_{i_2}\dots B_{i_s} : s > 0, B_{i_j} \in \mathcal{B}, B_{i_1} \geq B_{i_2} \geq \dots B_{i_s}\}$ is a basis, called the Poincaré-Birckhoff-Witt basis, of the universal enveloping algebra of $L_{\mathbf{k}}(\mathcal{X})$ which in turn may be identified with $A_{\mathbf{k}}(\mathcal{X})$. In a combinatorial setting the quest for formulas for the iterated integrals in the exponential product expansion amounts to searching for explicit formulae for the dual bases of such Poincaré-Birckhoff-Witt bases. In Melançon and Reutenauer [37] provided such an explicit formula for the dual basis for the PBW basis constructed on a Lyndon basis \mathcal{H} for the free Lie algebra $L(\mathcal{X})$. They recognized that the formula was virtually identical to the one obtained by Schützenberger [45] in 1958 for Hall-bases in the original narrow sense. Later work [38] showed that these formulae do indeed hold for the general Hall-Viennot bases \mathcal{H} as in (10). Modulo some minor notational changes, such as reading words from the left or from the right, the Melançon and Reutenauer's result is expressed in the following formula *à la Hopf algebra* [37]

$$\sum_{w \in W(\mathcal{X})} w \otimes w = \prod_{H \in \mathcal{H}} \exp([H] \otimes \beta^H) \quad (34)$$

where

$$\begin{aligned} \beta^a &= a \text{ if } a \in \mathcal{X}, \\ \beta^{Ma} &= \beta^M a \text{ if } H = Ma \in \mathcal{H}, a \in \mathcal{X}, M \in W(\mathcal{X}) \text{ and} \\ \beta^w &= \frac{1}{n_1!} \dots \frac{1}{n_s!} (\beta^{H_1}) \sqsupset^{n_1} \sqsupset \dots \sqsupset (\beta^{H_1}) \sqsupset^{n_1} \\ &\text{if } w = H_1^{n_1} \dots H_s^{n_s}, \text{ with } H_j \in \mathcal{H}, \text{ and } H_1 > \dots > H_s. \end{aligned} \quad (35)$$

In the special case of a Hall word $H = H_1 \dots H_s a$ with $a \in \mathcal{X}$, $H_1 > \dots > H_s < a$ with $H_i \in \mathcal{H}$ one obtains

$$\beta^H = \beta^{H_1} \sqsupset \beta^{H_2} \sqsupset \dots \sqsupset \beta^{H_s} \cdot a \quad (36)$$

which is a remarkable mix of the shuffle product and one instance of the concatenation product. These two products are really at home in two different algebras which are their respective (algebraic and/or topological) duals (for details see [22, 44, 54]). The next section introduces an entirely different product which will beautifully resolve this flaw.

4 Chronological products

Chronological products go back at least as far as the work of Schützenberger [45]. In 1958 he used a product \top in the algebra $A(\mathcal{X})$ that satisfies the identity (42) and thus is a chronological product. Incidentally, while never giving a name to this product, Schützenberger defined the shuffle product, which he denoted by $\overline{\top}$, as the symmetrization $w\overline{\top}z = w\top z + z\top w$ of the chronological product. The term *chronological product* was coined later and introduced into nonlinear control theory by Agrachev and Gamkrelidze who developed a comprehensive chronological calculus starting in the late 1970s [1, 2]. The chronological product also was implicitly a key ingredient in Sussmann’s exponential product expansion (15) - (18). In this section we shall make the case that the chronological product is the most natural product of nonlinear control. The subsequent section will give a more abstract foundation and more theoretical properties of the chronological algebra.

Probably the most basic, best known, and best studied example, e.g. [4], to illustrate the fundamental concepts of geometric nonlinear control is the system

$$\begin{aligned} \dot{x}_1 &= u_1 & |u_1| &\leq 1 \\ \dot{x}_2 &= u_2 & |u_2| &\leq 1 . \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{37}$$

This system is completely controllable (every point can be reached from every other point), since the distribution spanned by the vector fields $f_1 = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}$ and $f_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$ is nonintegrable due to the nonvanishing commutator $[f_1, f_2] = 2 \frac{\partial}{\partial x_3}$. To more clearly see the structure perform the global coordinate change $y_1 = x_1$, $y_2 = x_2$, and $y_3 = (x_3 + x_1 x_2)/2$ to rewrite the system in the less symmetric, but simpler form

$$\begin{aligned} \dot{y}_1 &= u_1 & |u_1| &\leq 1 \\ \dot{y}_2 &= u_2 & |u_2| &\leq 1 . \\ \dot{y}_3 &= y_1 u_2 \end{aligned} \tag{38}$$

It is convenient to consider the function

$$u_3(t) = (u_1 \star u_2)(t) \stackrel{\text{def}}{=} \left(\int_0^t u_1(s) ds \right) \cdot u_2(t) \tag{39}$$

a *virtual* third control which allows one to steer the system into an independent third direction corresponding to the commutator $[f_1, f_2] = 2 \frac{\partial}{\partial x_3}$. Via highly oscillatory controls this concept gives rise to path-planning algorithms [28, 29]. Such virtual control

also lend themselves to be used in cost-functionals to arrive at nonlinear optimal feedback stabilization schemes. This concept of the *virtual controls* may be considered a starting point to motivate a possible chronological product $(u_1 \star u_2)$ of two control functions u_1 and u_2 . If the functions u_1 and u_2 are both locally integrable, then so is the product $(u_1 \star u_2)$. Also note that the symmetrization $(u_1 \star u_2) + (u_2 \star u_1)$ corresponds to the derivative of the pointwise product of the integrated controls

$$(u_1 \star u_2 + u_2 \star u_1)(t) = \frac{d}{dt} \left(\left(\int_0^t u_1(s) ds \right) \left(\int_0^t u_2(s) ds \right) \right) \quad (40)$$

Finally, one easily verifies that this product satisfies a three term identity

$$\begin{aligned} (u_1 \star (u_2 \star u_3))(t) &= \left(\int_0^t u_1(s) ds, ds \right) \cdot \left(\left(\int_0^t u_2(s) ds \right) \cdot u_3(t) \right) \\ &= \left(\int_0^t \left(\int_0^s u_1(\sigma) d\sigma \right) \cdot u_2(s) ds \right) \cdot u_3(t) \\ &\quad + \left(\int_0^t \left(\int_0^s u_2(\sigma) d\sigma \right) \cdot u_1(s) ds \right) \cdot u_3(t) \\ &= ((u_1 \star u_2) \star u_3)(t) + ((u_2 \star u_1) \star u_3)(t) \end{aligned} \quad (41)$$

In general define a (right) *chronological algebra* to be a linear space C (over a field \mathbf{k}) that is endowed with a bilinear product $*: C \times C \mapsto C$ which satisfies the (right) *chronological identity*

$$v * (w * z) = (v * w) * z + (w * v) * z \quad \text{for all } v, w, z \in C \quad (42)$$

Thus the space $\mathcal{L}_{\text{loc}}([0, \infty))$ of locally integrable functions defined on the interval $[0, \infty)$ is a (right) chronological algebra with the product \star defined as above. In the following it turns out to often be more convenient to work with the integrated controls which *live* in the space $\mathcal{AC}_{\text{loc}}([0, \infty))$ of locally absolutely continuous functions. On this space define closely related product by

$$(U_1 * U_2)(t) \stackrel{\text{def}}{=} \left(\int_0^t U_1(s) U_2'(s) ds \right) \quad (43)$$

where $U'(s) = \frac{d}{ds} U(s)$ denotes the derivative. One immediately verifies that

$$\begin{aligned} (F * (G * H))(t) &= \int_0^t F(s) \cdot \left(\frac{d}{ds} \int_0^s G(\sigma) H'(\sigma) d\sigma \right) ds \\ &= \int_0^t F(s) G(s) \cdot H'(s) ds \\ &= \int_0^t \left(\int_0^s F(\sigma) G'(\sigma) d\sigma + \int_0^s G(\sigma) F'(\sigma) d\sigma \right) \cdot H'(s) ds \\ &= ((F * G) * H)(t) + ((F * G) * H)(t) \end{aligned} \quad (44)$$

is true for all $F \in \mathcal{AC}_{\text{loc}}$ and $G, H \in \mathcal{AC}_{\text{loc},0}$. Hence the space $\mathcal{AC}_{\text{loc},0}([0, \infty))$ of locally absolutely continuous functions that vanish at zero is (right) chronological algebra with the product $*$. It is the requirement $\int_0^t (FG)' = (FG)(t)$ that stands in the way of extending $*$ to $\mathcal{AC}_{\text{loc}} \times \mathcal{AC}_{\text{loc}}$ without giving up the chronological identity. (This is intimately related to the impossibility to equip any chronological algebra with an identity that satisfies $1 * 1 = 1$, compare (58)). However, it is convenient to still use that $U * 1 = 0$ for any U , and $1 * U = U$ for $U \in \mathcal{AC}_{\text{loc},0}$. The symmetrization of the product $*$ of (43) yields the (associative) pointwise product on $\mathcal{AC}_{\text{loc},0}$ (which in turn corresponds to the shuffle product in the combinatorial setting).

$$(U_1 * U_2) + (U_2 * U_1) = (U_1 \cdot U_2) \text{ for } U_1, U_2 \in \mathcal{AC}_{\text{loc},0} \quad (45)$$

There are many familiar chronological subalgebras of the ones defined above, most notably those of polynomial and of (real, or complex) exponential functions. In either case the multiplication rules are

$$\begin{aligned} x^n \star x^m &= \frac{1}{n+1} x^{n+m+1} & e^{nt} \star e^{mt} &= \frac{1}{n} e^{(n+m)t} \\ x^n * x^m &= \frac{m}{n+m} x^{n+m} & e^{nt} * e^{mt} &= \frac{m}{n+m} e^{(n+m)t} \end{aligned} \quad (46)$$

The standard method of solving time-varying linear differential equations by iteration is efficiently encoded by the chronological product: The integrated form of the initial value problem, the universal control system (13) $\dot{S} = S \cdot \Phi$, $\Phi(0) = 1$ with $\Phi = \sum_{i=1}^m u^i X_i$ may be very compactly written using chronological products

$$S = 1 + S * \Phi \quad (47)$$

Iteration immediately yields an explicit series expansion (respecting the noncommutative character of the variables X_j *inside* Φ)

$$\begin{aligned} S &= 1 + (1 + S * \Phi) * \Phi \\ &= 1 + \Phi + ((1 + S * \Phi) * \Phi) * \Phi \\ &= 1 + \Phi + (\Phi * \Phi) + ((1 + S * \Phi) * \Phi) * \Phi \\ &\vdots \\ &= 1 + \Phi + (\Phi * \Phi) + ((\Phi * \Phi) * \Phi) + (((\Phi * \Phi) * \Phi) * \Phi) \dots \end{aligned} \quad (48)$$

Upon expansion this immediately yields a suggestive formula for the iterated integrals (6) of the Chen Fliess series. For a word $w \in W(\mathcal{X})$ and a letter $a \in \mathcal{X}$

$$\Upsilon^{wa}(t, u) = \int_0^t \Upsilon^w(s, u) \cdot \left(\frac{d}{ds} \Upsilon^a(s, u) \right) ds = (\Upsilon^w(\cdot, u) * \Upsilon^a(\cdot, u))(t) \quad (49)$$

Similarly, the formula for the iterated integrals (12) in the exponential product expansion (9) simplifies dramatically when written in terms of chronological products

$$C^{HK}(t, u) = C^H(t, u) * C^K(t, u) \quad \text{if } H, K, HK \in \mathcal{H} \quad (50)$$

Recall that the original formula (12) contains factorials that correspond to stationary subsequences of the nonincreasing sequences of Hall words in the unique factorization (31). It is by virtue of the identity (65) that the chronological product absorbs these factorials that are necessary when using the shuffle product (which corresponds to pointwise multiplication of the iterated integrals (27)).

Chronological products also provide the most compact way for specifying a normal form for *free nilpotent control systems of rank r* . The key is to index the coordinates by Hall words $H \in \mathcal{H}^{(r)} \stackrel{\text{def}}{=} \{H \in \mathcal{H}: |h| \leq r\}$ (rather than by the integers):

$$\begin{aligned} \dot{x}_a &= u_a && \text{if } a \in \mathcal{X} \\ x_{HK} &= x_H * x_K && \text{if } H, K, HK \in \mathcal{H}^{(r)}(\mathcal{X}) \end{aligned} \quad (51)$$

For a typical Hall set on the alphabet $\mathcal{X} = \{0, 1\}$ and $r = 5$ the normal form for a free nilpotent system is readily obtained from (51) as

$$\begin{aligned} \dot{x}_0 &= u_0 \\ \dot{x}_1 &= u_1 \\ \dot{x}_{01} &= x_0 \cdot \dot{x}_1 = x_0 u_1 \\ \dot{x}_{001} &= x_0 \cdot \dot{x}_{01} = x_0^2 u_1 && \text{using } \psi^{-1}(001) = (0(01)) \\ \dot{x}_{101} &= x_1 \cdot \dot{x}_{01} = x_1 x_0 u_1 && \text{using } \psi^{-1}(101) = (1(01)) \\ \dot{x}_{0001} &= x_0 \cdot \dot{x}_{001} = x_0^3 u_1 && \text{using } \psi^{-1}(0001) = (0(0(01))) \\ \dot{x}_{1001} &= x_1 \cdot \dot{x}_{001} = x_1 x_0^2 u_1 && \text{using } \psi^{-1}(1001) = (1(0(01))) \\ \dot{x}_{1101} &= x_1 \cdot \dot{x}_{101} = x_1^2 x_0 u_1 && \text{using } \psi^{-1}(1101) = (1(1(01))) \\ \dot{x}_{00001} &= x_0 \cdot \dot{x}_{0001} = x_0^4 u_1 && \text{using } \psi^{-1}(00001) = (0(0(0(01)))) \\ \dot{x}_{10001} &= x_1 \cdot \dot{x}_{0001} = x_1 x_0^3 u_1 && \text{using } \psi^{-1}(10001) = (1(0(0(01)))) \\ \dot{x}_{11001} &= x_1 \cdot \dot{x}_{1001} = x_1^2 x_0^2 u_1 && \text{using } \psi^{-1}(11001) = (1(1(0(01)))) \\ \dot{x}_{01001} &= x_{01} \cdot \dot{x}_{001} = x_{01} x_0^3 u_1 && \text{using } \psi^{-1}(01001) = ((01)(0(01))) \\ \dot{x}_{01101} &= x_{01} \cdot \dot{x}_{101} = x_{01} x_1^2 x_0 u_1 && \text{using } \psi^{-1}(01101) = ((01)(1(01))) \end{aligned} \quad (52)$$

The coordinates in this normal form are chosen such that there are no factorials in this system description - consequently the values of $x_H(t, u)$ and the iterated integrals $\beta_H(t, u)$ of the exponential product expansion will disagree by multi-factorials.

It is straightforward to extract from this formula (51) the components $\langle dx_H, f_a \rangle$ of the system vector fields f_a , $a \in \mathcal{X}$ in these coordinates. Specifically, rewrite the system

(51) in the form

$$\dot{x} = \sum_{a \in \mathcal{X}} u^a f_a(x). \quad (53)$$

and find that

$$f_a = \sum_{H \in \mathcal{H}^{(r)}} x_{Ha^{-1}} \frac{\partial}{\partial x_H} \quad (54)$$

where $(wb)a^{-1} = w$ if $a = b$ and $(wb)a^{-1} = 0$ else (for any $w \in W(\mathcal{X})$ and $b \in \mathcal{X}$). Moreover, if $w = H_1 H_2 \dots H_s$ is the unique factorization (31) of a word $w \in W(\mathcal{X})$ into a nonincreasing product of Hall words, then define $x_w = x_{H_1} x_{H_2} \dots x_{H_s}$. The Lie algebra generated by the system vector fields f_a , $a \in \mathcal{X}$ is, by construction, free nilpotent of rank r . This also may be verified by direct calculation. Incidentally, Grayson and Grossman [14] first proposed essentially the formula (54) and then verified by direct calculation that the associated Lie algebra was free – thereby obtaining essentially the formulas for the iterated integrals of the exponential product expression (9) through another independent route.

5 The free chronological algebra

This section provides a more formal treatment of chronological algebras and derives some simple, but remarkable identities. Following [22] the free chronological algebra is presented as a chronological algebra of iterated integral functionals, and the Chen Fliess series is recognized as the image of the resolution of the identity map under a natural isomorphism.

On the spaces $A_{\mathbf{k},0}(\mathcal{X})$ and $\hat{A}_{\mathbf{k},0}(\mathcal{X})$ of noncommutative polynomials and power series with zero constant term, define a bilinear, noncommutative, nonassociative product $*$ by identifying $w * a = wa$ for any nonempty word $w \in W_0(\mathcal{X})$ and any letter $a \in \mathcal{X}$, and using the chronological identity (42) to *define*

$$w * (z * a) = (w * z) * a + (z * w) * a \quad \text{for } a \in \mathcal{X}, w, z \in W_0(\mathcal{X}) \quad (55)$$

One easily sees that with this product $\mathcal{C}_{\mathbf{k}}(\mathcal{X}) \stackrel{\text{def}}{=} (A_{\mathbf{k},0}(\mathcal{X}), *)$ is a chronological algebra that is *free* in the usual sense: If C is any chronological algebra then any $\gamma: \mathcal{X} \mapsto C$ extends uniquely to a chronological algebra homomorphism $\hat{\gamma}: \mathcal{C}_{\mathbf{k}}(\mathcal{X}) \mapsto C$.

Utilizing the recursive definition of the shuffle product (25), one recognizes that the shuffle product is (the extension to $A_{\mathbf{k}}(\mathcal{X})$ of) the symmetrization of the chronological product:

$$w * z + z * w = w \sqcup z \quad \text{for } w, z \in W_0(\mathcal{X}) \quad (56)$$

which implies in particular that

$$w * (z * a) = (w \sqcup z)a \quad \text{for } a \in \mathcal{X}, w, z \in W_0(\mathcal{X}) \quad (57)$$

Note that the special case of $w \sqcup w = 2w * w$ clearly indicates that one cannot extend the chronological product to the empty word without giving up some fundamental property. For notational convenience one may extend $*$ to $A_{\mathbf{k}}(\mathcal{X}) \times A_{\mathbf{k},0}(\mathcal{X}) \cup A_{\mathbf{k},0}(\mathcal{X}) \times A_{\mathbf{k}}(\mathcal{X})$ by declaring

$$1 * w = w \text{ for } w \in W_0(\mathcal{X}) \text{ and } w * 1 = 0 \text{ for } w \in W(\mathcal{X}). \quad (58)$$

One readily verifies that the identities (42), (56), and (57) are still satisfied.

As an example consider the chronological product of two two-letter words

$$\begin{aligned} (ab) * (cd) &= ((a * b) * c) * d + (c * (a * b)) * d \\ &= ((a * b) * c) * d + ((c * a) * b) * d + ((a * c) * b) * d. \end{aligned} \quad (59)$$

which includes the special case $(ab) * (ab) = abab + 2aabb = \frac{1}{2}(ab)\sqcup(ab)$. This example leads to a number of identities that relate the different products (concatenation, shuffle and chronological) on the set $A_{\mathbf{k},0}(\mathcal{X})$. Due to the lack of associativity one needs to distinguish between left and right chronological powers: For $w \in A_{\mathbf{k},0}(\mathcal{X})$ define $w^{*1} = \lambda^1(w) = w^{\sqcup 1} = w$, and inductively for $n \geq 1$

$$\lambda^{n+1}(w) = w * \lambda^n(w) \quad (60)$$

$$w^{*(n+1)} = w^{*n} * w \quad (61)$$

$$w^{\sqcup(n+1)} = w \sqcup w^{\sqcup n} = w^{\sqcup n} \sqcup w \quad (62)$$

(allowing $w \in A_{\mathbf{k}}(\mathcal{X})$ for the shuffle powers). One obtains the useful identities for $w \in A_{\mathbf{k},0}(\mathcal{X})$ for $n > 1$

$$w * w^{*(n-1)} = (n-1) \cdot w^{*n} \quad (63)$$

$$\lambda^n(w) = (n-1)! \cdot w^{*n} \quad (64)$$

$$w^{\sqcup n} = n! \cdot w^{*n} \quad (= n\lambda^n(w)) \quad (65)$$

In the case of $n = 2$ note that $w^{\sqcup 2} = w \sqcup w = w * w + w * w = 2w^{*2} = 2\lambda^2(w)$. The case $n = 3$ is just the chronological identity (42) with $v = w = z$: $\lambda^3(w) = w * w^{*2} = 2w^{*3}$. In general, for $n \geq 3$, one obtains by induction

$$\begin{aligned} w * w^{*(n-1)} &= w * (w^{*(n-2)} * w) \\ &= (w * w^{*(n-2)}) * w + (w^{*(n-2)} * w) * w \\ &= (n-2) \cdot w^{*(n-1)} * w + w^{*n} \\ &= (n-1) \cdot w^{*n} \end{aligned} \quad (66)$$

and as corollaries for $n \geq 3$

$$\lambda^n(w) = w * (\lambda^{n-1}(w)) = (n-2)! \cdot w * (w^{*(n-1)}) = (n-1)! \cdot w^{*(n-1)} \quad (67)$$

and

$$\begin{aligned} w^{\sqcup n} = w \sqcup w^{\sqcup(n-1)} &= w \sqcup \left((n-1)! w^{*(n-1)} \right) \\ &= (n-1)! \cdot \left(w * w^{*(n-1)} + w^{*(n-1)} * w \right) \\ &= (n-1)! \cdot ((n-1) + 1) \cdot w^{*n} \\ &= n! \cdot w^{*n} \end{aligned} \quad (68)$$

Returning to the exponential product expansion (9) recall that the multi-factorials disappear when the iterated integrals are written in terms of chronological powers (50). The multi factorials are ever present when using pointwise multiplication of the iterated integrals (12), when using shuffle products for the dual PBW bases (36), or when differentiating the normal forms of free nilpotent systems (20).

Specifically, rewrite the formulae (36) for the dual PBW bases of Melançon and Reutenauer [37] using chronological powers: If $H = H_1^{n_1} \dots H_s^{n_s} a$ is the unique standard factorization of a Hall word then the corresponding dual basis element is with factorials

$$\beta^H = \frac{1}{n_1!} \dots \frac{1}{n_s!} \left(\underbrace{\beta^{H_1} * \dots * \beta^{H_1}}_{n_1\text{-times}} * \dots * \underbrace{\left(\beta^{H_s} * \dots * \left(\beta^{H_s} * \beta^a \right) \right)}_{n_s\text{-times}} \dots \right) \quad (69)$$

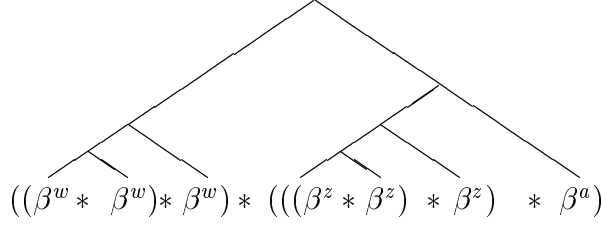
and without factorials

$$\beta^H = \left(\left(\beta^{H_1} \right)^{*n_1} * \left(\left(\beta^{H_2} \right)^{*n_2} * \dots * \left(\left(\beta^{H_s} \right)^{*n_s} * a \right) \dots \right) \right) \quad (70)$$

In either case the uniform chronological products replace the mix of shuffle and concatenation products. However, the disappearance of the multi-factorials comes at the cost of a reversal of the bracketing of repeat factors, illustrated for the case of $H = w w w z z z a$ with $w = H_1 > z = H_2 \in \mathcal{H}$, $a \in \mathcal{X}$. In this case

$$\begin{aligned} \beta^H &= \frac{1}{36} (\beta^w * (\beta^w * (\beta^w * (\beta^z * (\beta^z * (\beta^z * \beta^a)))))) \\ &= (((\beta^w * \beta^w) * \beta^w) * (((\beta^z * \beta^z) * \beta^z) * \beta^a) \\ &= ((\beta^w)^{*3} * ((\beta^w)^{*3} * \beta^a)) \end{aligned} \quad (71)$$

Pictorially:



However, for the dual PBW bases elements of words $w \in W(\mathcal{X}) \setminus \mathcal{H}$ that are not Hall words, the noncommutative chronological product requires an explicit summation that is implicit in the formula (35) using shuffle products. Specifically, if $w = H_1^{n_1} \dots H_r^{n_r}$ is the standard factorization of $w \in W(\mathcal{X})$ with $H_i \in \mathcal{H}$, $H_i > H_{i+1}$ then

$$\beta^w = \sum_{\sigma \in S_r} \left((\beta^{H_{\sigma(1)}})^{*n_{\sigma(1)}} * \left(\dots \left((\beta^{H_{\sigma(r-1)}})^{*n_{\sigma(r-1)}} * (\beta^{H_{\sigma(r)}})^{*n_{\sigma(r)}} \right) \dots \right) \right) \quad (72)$$

where the sum is taken over all permutations σ of the set $\{1, 2, \dots, r\}$.

For letters $a \in \mathcal{X}$ define the maps ς_a and ρ_a on $A_{\mathbf{k}}(\mathcal{X})$ as transposes of the left and right translations $w \mapsto aw$ and $w \mapsto wa$, i.e. $\varsigma(1) = \rho(1) = 0$ and for $w \in W(\mathcal{X})$ and $b \in \mathcal{X}$ by

$$\begin{aligned} \varsigma_a(bw) &= w & \text{if } a = b, & \quad \text{and} \quad \varsigma_a(bw) = 0 & \text{else} \\ \rho_a(wb) &= w & \text{if } a = b, & \quad \text{and} \quad \rho_a(wb) = 0 & \text{else} \end{aligned} \quad (73)$$

Using the recursive definition (25) and the commutativity of the shuffle product one easily sees that both ς_a and ρ_a are derivations on $(\hat{A}_{\mathbf{k}}(\mathcal{X}), \mathbb{W})$ for $a \in \mathcal{X}$. However, using (55), one sees that on the chronological algebra $\mathcal{C}_{\mathbf{k}}(\mathcal{X}) = (A_{\mathbf{k}}(\mathcal{X}), *)$ only ς_a is a derivation (for $a \in \mathcal{X}$) (where defined, and using the extended convention (58) to resolve the special cases). Specifically, for $w, z \in A_{\mathbf{k},0}(\mathcal{X})$, and $a, b \in \mathcal{X}$

$$\begin{aligned} \varsigma_a(w * (zb)) &= \varsigma_a((w * z + z * w)b) \\ &= (\varsigma_a(w \mathbb{W} z))b \\ &= (\varsigma_a(w)) * (zb) + w * (\varsigma_a(zb)) \end{aligned} \quad (74)$$

while the right truncation ρ_a is not a derivation since, e.g. for distinct letters $a, b, c, d \in \mathcal{X}$, $\rho_b(ab * cd) = 0$ (compare (59)), whereas $(\rho_b(ab)) * (cd) + (ab) * (\rho_b(cd)) = acd + cad \neq 0$. Instead one has (for $w, z \in A_{\mathbf{k},0}(\mathcal{X})$, and $a, b \in \mathcal{X}$)

$$\rho_a(w * (zb)) = w * (\rho_a(zb)). \quad (75)$$

This property of being only a one-sided derivation plays a fundamental role in the theoretical development of presentations of the free chronological algebra [22]. Note, that

in general the product (composition) of two derivations is not a derivation, but their commutator always is a derivation. Consequently, one obtains well-defined derivations ς_ℓ and ρ_ℓ on $(A_{\mathbf{k}}(\mathcal{X}), \mathbb{W})$ for $\ell \in \hat{L}_{\mathbf{k}}(\mathcal{X})$.

All these formal objects have natural analytic interpretations, compare [22] for details: Recall that by virtue of Ree's theorem (30) the set $\hat{G}_{\mathbf{k}}(\mathcal{X}) \subseteq \hat{A}_{\mathbf{k}}(\mathcal{X})$ of exponential Lie series has a natural Lie group structure (when endowed with the usual topology). Considering the exponential Lie series as formal *points*, the algebra $(A_{\mathbf{k}}(\mathcal{X}), \mathbb{W})$ is interpreted as an algebra of *functions* on $\hat{G}_{\mathbf{k}}(\mathcal{X})$ with pointwise multiplication by virtue of (30). Specifically, an element $w \in A_{\mathbf{k}}(\mathcal{X})$ is identified with the linear functional $w: \Phi \mapsto \langle \Phi, w \rangle$ on $\hat{G}_{\mathbf{k}}(\mathcal{X})$. The formal *tangent vectors* to $\hat{G}_{\mathbf{k}}(\mathcal{X})$ at a point $\Phi \in \hat{G}_{\mathbf{k}}(\mathcal{X})$ are the linear functionals $\xi_\Phi \in \hat{A}_{\mathbf{k}}(\mathcal{X})$ that satisfy $\xi_\Phi(w \mathbb{W} z) = (\xi_\Phi(w)) \cdot \langle \Phi, z \rangle + \langle \Phi, w \rangle \cdot (\xi_\Phi(z))$ for all $w, z \in A_{\mathbf{k}}(\mathcal{X})$. One easily verifies, compare [22], that any such ξ_Φ is of the form $\xi_\Phi = \Phi \ell$ for some element $\ell \in \hat{L}_{\mathbf{k}}(\mathcal{X})$. A simple calculation makes the connection between formal tangent vectors and the right truncations ρ_ℓ

$$\begin{aligned}
\langle \Phi \ell, (w \mathbb{W} z) \rangle &= \langle \Delta(\Phi \ell), (w \otimes z) \rangle \\
&= \langle (\Phi \otimes \Phi)(1 \otimes \ell + \ell \otimes 1), (w \otimes z) \rangle \\
&= \langle \Phi \otimes \Phi, (\rho_\ell(w) \otimes z) + (w \otimes \rho_\ell(z)) \rangle \\
&= \langle \Phi, (\rho_\ell(w) \mathbb{W} z) + (w \mathbb{W} \rho_\ell(z)) \rangle
\end{aligned} \tag{76}$$

Finally, any $\ell \in \hat{L}_{\mathbf{k}}(\mathcal{X})$ may also be identified with the map $\ell: \Phi \mapsto \Phi \ell$ ($\Phi \in \hat{G}_{\mathbf{k}}(\mathcal{X})$) which is interpreted as a *left invariant tangent vector field* on $\hat{G}_{\mathbf{k}}(\mathcal{X})$. Similarly one may consider right invariant vector fields which are related to the left truncations ς_ℓ with $\ell \in \hat{L}_{\mathbf{k}}(\mathcal{X})$.

Following [22] one may present the free chronological algebra $\mathcal{C}_{\mathbf{k}}(\mathcal{X})$ as an *chronological algebra of dynamic functionals*. This is in complete analogy to the presentation of the associative polynomial algebra $\mathbf{k}[X_1, \dots, X_n]$ in commuting variables X_i by $\mathbf{k}[x_1, \dots, x_n]$ of polynomial functions as the subalgebra of $\mathbf{k}^{(\mathbf{k}^m)}$ that is generated by the projections $x_i: (p_1, \dots, p_n) \mapsto p_i$, $i = 1, \dots, n$. In the chronological case, one may for example, choose $\mathcal{U} = \mathcal{A}\mathcal{C}_{\text{loc},0}([0, \infty))$ (the algebra of locally absolutely continuous functions that vanish at zero) as ring of coefficients. Then $\mathcal{U}^{(\mathcal{U}^{\mathcal{X}})}$ is a chronological algebra under pointwise multiplication. Define $\mathcal{I}\mathcal{I}\mathcal{F}_{\mathbf{k}}(\mathcal{X})$ as the chronological subalgebra of $\mathcal{U}^{(\mathcal{U}^{\mathcal{X}})}$ that is generated by the projections $\pi_a: \mathcal{U}^{\mathcal{X}} \mapsto \mathcal{U}$ defined by $\pi_a(U) = U^a$ for $a \in \mathcal{X}$. Denote by Υ the map $A_{\mathbf{k}}(\mathcal{X}) \mapsto \mathcal{I}\mathcal{I}\mathcal{F}_{\mathbf{k}}(\mathcal{X})$ that sends the generators $a \in \mathcal{X}$ to $\Upsilon(a) = \pi^a$. As is shown in [22] the map Υ is not only surjective, but also a

chronological algebra isomorphism. The proof relies on the choice of $\mathcal{A}_{\text{loc},0}([0, \infty))$ which is a sufficiently large set of *dynamic scalars*, and on the fact that ρ_a is not a derivation on the chronological algebra. Specifically, for any *given* set $\{f_a \in A_{\mathbf{k}}(\mathcal{X}) : a \in \mathcal{X}\}$ there is a unique $g \in A_{\mathbf{k},0}(\mathcal{X})$, (namely $g = \sum_{a \in \mathcal{X}} f_a a$) such that $\rho_a(g) = f_a$ for each $a \in \mathcal{X}$.

Finally we return to the Chen-Fliess series which appears in a new light in this advanced context. Recall the infinite product expansion *à la Hopf algebra* (34) of [37]. On the left hand side we find the sum $\sum_{w \in W(\mathcal{X})} w \otimes w$ which may be interpreted as the identity map on $A_{\mathbf{k}}(\mathcal{X})$, under the standard identification of the the space of linear maps $\text{Hom}(V, V)$ on a vector space V with the tensor product $V^* \otimes V$ (here $V = A_{\mathbf{k}}(\mathcal{X})$ and $V^* = \hat{A}_{\mathbf{k}}(\mathcal{X})$ is its algebraic dual). Thus the Chen-Fliess-series (5) is nothing else than the image of the identity map on $A_{\mathbf{k}}(\mathcal{X})$, identified with $\sum_{w \in W(\mathcal{X})} w \otimes w \in \hat{A}_{\mathbf{k}}(\mathcal{X}) \otimes A_{\mathbf{k}}(\mathcal{X})$ under the map $\text{id}_{\hat{A}_{\mathbf{k}}(\mathcal{X})} \otimes \Upsilon$. On either side one may go from the infinite series to the exponential product expansion. The two product expansions are mapped into each other by the same map $\text{id}_{\hat{A}_{\mathbf{k}}(\mathcal{X})} \otimes \Upsilon$ [22].

6 Concluding remarks

This article has laid out how the algebraic structure of chronological algebras perfectly matches the geometric features of noncommuting flows that are at the heart of affine control systems. This chronological structure is useful for describing geometric features, for analysis (controllability, optimality, approximating systems), and for design (including stabilization and path planning). The parallels between the combinatorial setting and the dynamical systems point of view may be exploited in several ways: For proofs, calculations, and for obtaining algebraic and geometric insight into specific features and properties, one has the choice between combinatorial and differential equations approaches. The realization of the the abstract chronological algebra in terms of integral functionals and dynamic variables makes the abstract algebra *tangible*. In the other direction, the combinatorial language allows one to dispense of the excessive notation that used to burden manipulations of iterated integrals: The isomorphism Υ allows one to instead only manipulate the indices.

The chronological algebra structure has recently found substantial attention also outside nonlinear control theory: Following the pioneering work of Crouch and Grossman [9] who saw the applicability of the framework of noncommuting flows to the numerical solution of systems of differential equations under state constraints, several numerical

analysts have been exploring a very closely related formalism, compare e.g. [39]. In the completely different arena of homological algebra, in particular Loday [30, 31, 32, 33], has analyzed the algebraic structure of Leibniz algebras. These are characterized by the three-term identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \text{ for all } x, y, z \in \mathfrak{g} \quad (77)$$

As such a Leibniz algebra is like a Lie algebra but without the requirement of anticommutativity. As spelled out in [31], the study of Leibniz algebras may be motivated by the observation that the identity (77) alone is sufficient to prove as basic properties as $d^2 = 0$ for the boundary map d of the complex

$$C_*(\mathfrak{g}) = \dots \mathfrak{g}^{\otimes n} \xrightarrow{d} \mathfrak{g}^{\otimes n-1} \longrightarrow \dots \longrightarrow \mathfrak{g} \longrightarrow \mathbf{k} \quad (78)$$

for a Lie algebra \mathfrak{g} . The Leibniz algebra, as a *nonabelian* version of a Lie algebra, is intimately related to the chronological algebra which may be regarded as a nonassociative pre-shuffle algebra. Technically the Leibniz algebra is a dual algebra structure to that of the chronological algebra [33]. For a comprehensive introduction to the notion of duality, specifically Koszul duality of quadratic operads, we refer the interested reader to [13]. Incidentally, the original work of Agrachev and Gamkrelidze [1, 2] used the term *chronological* for what Loday calls Leibniz, and what we refer to as the dual of the chronological algebra.

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