Stabilization of Nonlinear Systems in the Plane

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Abstract. It is shown that every small-time locally controllable system in the plane can (locally) be asymptotically stabilized by employing locally Hölder continuous feedback laws, as essentially was conjectured by E. Sontag. An explicit algorithm for the construction of such feedback laws is given.

Keywords: Stabilization, controllability, nonlinear control, Hölder continuous feedback, Lyapunov function.

1 Introduction

This paper presents some recent advances in the design of nonlinear state-feedback stabilization schemes. For the clarity of the underlying arguments we restrict our considerations to single-input systems in the plane of the form

$$\dot{x} = f(x) + ug(x) \qquad f(0) = 0,\ g(0) \neq 0 \tag{1}$$

with f and g smooth (real analytic) vector fields on $\mathbf{R}^2$ and the control an integrable function with values in $\mathbf{R}$.

While over the past few years a tremendous progress has been made towards the understanding of (small-time) local controllability (STLC) of such systems on general n-dimensional space (e.g. [8, 10, 13, 15]) (i.e. under which conditions one can reach in arbitrary small positive time a neighbourhood of the initial point), it also has become clear how much more complicated the problem of feedback stabilization is (for n-dimensional systems), i.e. under which conditions can one find a stabilizing state-feedback law $u(x)$ such that the resulting closed-loop system

$$\dot{x} = f(x) + u(x)g(x) \tag{2}$$

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has $x^0 = 0$ as an (asymptotically) stable equilibrium point.

From the many results on feedback stabilization which have been found in recent years (see e.g. [1, 2, 4, 5, 6, 9, 11, 12, 14] and the references therein) we here only want to recall the two cornerstone results which show most clearly where our works fits in. The first guarantees under (relatively mild) controllability assumptions the existence of piecewise analytic stabilizing feedback laws, the other one gives very restrictive necessary conditions for the existence of stabilizing C^1-feedback laws. For details and the precise definitions of the employed notions we refer the reader to the original papers.

Theorem 1.1 (Sussmann [14]): *If the system $\dot{x} = f(x, u)$ on the manifold M^n is completely controllable (i.e. every point in the state-space can be reached from every other point in the state-space) then for every point $x^0 \in M^n$ there exists a piecewise analytic feedback controller for the system (1) which steers all of M^n into x^0.* (For the precise definition of a piecewise analytic vector fields, employing analytic stratifications, see the original paper.)

Theorem 1.2 (Brockett [5]): *A necessary condition for the existence of a continuously differentiable asymptotically stabilizing feedback law for the system $\dot{x} = f(x, u)$ is that*
(i) the linearized system has no uncontrollable modes associated with eigenvalues whose real part is positive;
(ii) there exists a neighbourhood N of $x^0 = 0$ and for each $\xi \in N$ there is a control u_ξ steering the system from $x = \xi$ at $t = 0$ to $x = 0$ at $t = \infty$;
(iii) the map $(x, u) \longrightarrow f(x) + ug(x)$ is onto a neighbourhood of 0.

We remark that many of the obstructions to the existence of continuous stabilizing feedback laws (for even very simple controllable systems) are basically of topological nature, for examples see [2, 5, 14]. However, there is a big gap between piecewise analytic and C^1-feedback laws – and this is precisely the place where our new results fit in. Specifically, for every system in the plane that is (small-time) locally controllable we explicitly construct a locally Hölder continuous feedback law (compare definition 2.1) that asymptotically stabilizes the system at the origin. While we expect that such *sublinear* feedback laws will also be of great importance for higher dimensional systems, it is unlikely that a similarily simple algorithm for their construction can be found (compare section 4). One of the crucial observations in this work is that one may employ continuous feedback laws that are not everywhere Lipschitz without having to give up the requirements that the resulting closed loop system has unique solutions forward in time (provided the feedback laws satisfy a natural transversality condition). For most practical applications one really is interested in forward uniqueness of solutions only, and thus in general we may allow solution curves to merge, and in particular, they may reach the equilibrium point in finite time (from the Lyapunov-like condition it will be clear that no solution will be able to leave the origin). Moreover, we are able to provide a constructive algorithm for finding such stabilizing feedback laws for systems in the plane.

Finally, the constructed feedback laws are also in certain ways compatible with standard nilpotent approximations (the details are still subject of further investigation): The Hölder exponents occurring in the feedback laws are expected to have a geometrical meaning that is closely related to that of the exponents of the minimum-time value function and the exponents in the standard nilpotent approximations.

2 Some preparatory technical results

In this section we establish conditions under which one can guarantee that the resulting closed-loop systems (which in general no longer have Lipschitz continuous right hand side) have unique solutions. Also, we show that under certain (essential) local controllability assumptions a naturally arising (and readily computable) Hamiltonian function is nonnegative with bounded sublevel sets so that it is a good candidate for a Ljapunov function for the closed-loop system.

We recall the following standard definition:

Definition 2.1 *A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is locally Hölder continuous of order $\alpha > 0$, if for every point $x_0 \in \mathbf{R}^n$ there are a neighbourhood U of x_0 and a constant $0 < C < \infty$ such that $|f(x_1) - f(x_2)| < C|x_1 - x_2|^\alpha$ for all $x_1, x_2 \in U$. (If $\alpha = 1$, then f is locally Lipschitz continuous.)*

Before we proceed with the special control systems under consideration, we need to verify the following modification of Gronwall's inequality:

Lemma 2.1 *If $\sigma : [0,T] \longrightarrow \mathbf{R}$ is continuously differentiable on $(0,T)$ and satisfies*

$$\sigma'(s) \leq \tfrac{M}{m} s^{-1+1/m} \sigma(s) \tag{3}$$

for $s \in (0,T)$ and positive constants M and m, then $\sigma(t) \leq e^{Mt^{1/m}} \sigma(0)$ for $t \in [0,T]$.

Proof: After multiplying both sides of (3) by $e^{-Ms^{1/m}}$ and rearranging terms we have

$$0 \geq \sigma'(s)e^{-Ms^{1/m}} - \sigma(s)\tfrac{M}{m}s^{-1+1/m}e^{-Ms^{1/m}} = \frac{d}{dt}\left(\sigma(t)e^{-Mt^{1/m}}\right)$$

and thus $\sigma(t)e^{-Mt^{1/m}}$ is nondecreasing over $[0,T]$. The assertion of the lemma is immediate.

Proposition 2.2 *For f and g locally Lipschitz continuous vector fields on $\mathbf{R}^2$, m an odd integer, and $\phi(x) = x_2^{1/m}$ consider the initial value problem*

$$\dot{x} = f(x) + \phi(x)g(x), \;\; x(0) = x^0 = (x_1^0, 0). \tag{4}$$

If U is a neighbourhood of x^0 such that f is transversal to $S \cap U = \{x : x_2 = 0\} \cap U$ at x^0 (i.e. $f_2(x^0) = <dx_2, f> (x^0) \neq 0$) then the initial value problem (4) *has a unique solution on the interval $(-t_1, t_1)$ for some $t_1 > 0$.*

Remark: This proposition can (modulo technicalities) easily be generalized to a setting of $\dot{x} = f(x) + \sum_{j=1}^{\rho} \phi_j(x) g^j(x),\ x(0) = x^0$ on a n-dimensional manifold M^n where the vector fields f and g^j are locally Lipschitz, and for each $j = 1, \ldots, \rho$ there is a positive integer m_j such that $(\phi_j(x))^{m_j}$ is continuously differentiable. If $S_j = \{x \in M : \phi_j(x) = 0\};\ j = 1, \ldots, \rho$ and if the bad set $S = \bigcup_{j=1}^{\rho} S_j$ is sufficiently nice, e.g. a locally finite union of embedded submanifolds of codimension at least one, and f is transversal to S at x^0, then uniqueness of solutions follows.

Since we here only need above special case, we leave the proof (and precise statement) of the generalized proposition to a fourthcoming paper on a general stabilization result (under preparation).

Proof: Since the differential equation in (4) has a continuous right hand side, solutions are guaranteed to exist for every initial condition. We proceed in two steps: First, by comparing solutions to (4) to those of the initial value problem with constant right hand side

$$\dot{x} = f(x^0),\ x(0) = x^0 \tag{5}$$

we show that every solution of the initial value problem (4) moves sufficiently fast away from the bad set $\{x : x_2 = 0\}$. In a second step we use the estimate obtained for the distance of solutions to (4) from the set $\{x : x_2 = 0\}$ to conclude that solutions are unique by applying lemma 2.1.

We write $f(x) = f_1(x)\frac{\partial}{\partial x_1} + f_2(x)\frac{\partial}{\partial x_2}$. For simplicity we assume that $f_1(x^0) = 0$; (this always can be achieved by the linear change of local coordinates $x_1^{\text{new}} = x_1^{\text{old}} - (f_1(x^0)/f_2(x^0))x_2^{\text{old}}$ and $x_2^{\text{new}} = x_2^{\text{old}}$). Clearly, for every fixed initial condition (5) has a unique solution: $\hat{x}(t) = (x_1^0, t f_2(x^0))$. Using that $\phi(x^0) = 0$ and the continuity of the right hand side of the differential equation in (4), for every $\varepsilon > 0$ there is an open neighbourhood $V_\varepsilon(x^0)$ of x^0 such that $|f(x) + \phi(x)g(x) - f(x^0)| < \varepsilon$ for all $x \in V_\varepsilon(x^0)$. Hence for any solution $\tilde{x}(\cdot)$ of (4) we obtain (e.g. compare [3], page 105, Theorem 3 with $L = 1$ as Lipschitz constant for (5)) $|\tilde{x}_2(t) - t f_2(x^0)| \leq \varepsilon(e^{|t|} - 1)$ for all $t > 0$ sufficiently small (i.e. such that both $\hat{x}(\cdot)$ and $\tilde{x}(\cdot)$ are inside $V_\varepsilon(x^0)$). Hence, for $|t|$ small we have $|\hat{x}_2(t)| \geq |t f_2(x^0) - \varepsilon(e^{|t|} - 1)| \geq C_1 t$ for all $|t|$ sufficiently small, and C_1 a positive constant (e.g., choose $0 < \varepsilon < |\frac{1}{4} f_2(x^0)|$.)

For each $\delta > 0$ let $U_\delta = \{x \in U : |x_2| \geq \delta\}$. Also, let L_1, L_2 be Lipschitz constants for f and g, respectively, on (the possibly shrunk neighbourhood) U, and B a bound for $|g|$ on

U. One obtains the Lipschitz constant $L_1 + L_2 \cdot 1 + B \cdot \frac{1}{m}\delta^{-1+1/m}$ for $F = f + \phi g$ on U_δ (assuming w.l.o.g. that $|x_2| \leq 1$ for all $x \in U$). Combined with the previous observation we find that for any two solutions $\bar{x}(t), \tilde{x}(t)$ of (4)

$$\begin{aligned} |F(\bar{x}(t)) - F(\tilde{x}(t)| &\leq (L_1 + L_2 + B \cdot \tfrac{1}{m}(C_1|t|)^{-1+1/m})|\bar{x}(t) - \tilde{x}(t)| \\ &\leq \tfrac{M}{m}|t|^{-1+1/m}|\bar{x}(t) - \tilde{x}(t)| \qquad \text{for some } M > 0. \end{aligned}$$

The assertion of the proposition follows after applying lemma 2.1 to the nonnegative function $\sigma(t) = \frac{1}{2}|\bar{x}(t) - \tilde{x}(t)|^2$, using that $\sigma(0) = 0$.

By a change of local coordinates (i.e. so that locally $g(x) \equiv \frac{\partial}{\partial x_1}$) and if necessary introduction of a preliminary smooth feedback ($u^{\text{new}} = u^{\text{old}} - f_1(x)$) (neither one affecting local controllability) one locally always may transform the general system (1) to the form

$$\left\{ \begin{array}{lcl} \dot{x}_1 &=& u \\ \dot{x}_2 &=& \hat{q}(x_1) + x_2 r(x_1, x_2) \end{array} \right. \tag{6}$$

with $\hat{q}$ and r analytic functions. We recall that for analytic single-input systems in the plane the Hermes-Condition [8] is necessary and sufficient for (small-time) local controllability; i.e. this system is STLC if and only if $\hat{q}(x_1) = x_1^{2p-1} q(x_1)$ with $q(0) \neq 0$ and p a positive integer. W.l.o.g. we may assume that $q(0) < 0$ (otherwise let $x_2^{\text{new}} = -x_2^{\text{old}}$). Define the function $H^\phi(x_1, x_2) = -\int(\hat{q}(x_1) + x_2 r(x_1, x_2))dx_1 + \phi(x_2)$ (with the constant of integration chosen such that $H(0, x_2) = \phi(x_2)$, $x_2 \in \mathbf{R}$. We claim the following:

Claim 2.3 *If $\phi(x_2) = E x_2^{\alpha+1}$ with $\alpha = 1/(2p-1)$ then H^ϕ is positive on a deleted neighbourhoood of the origin, provided the constant E is chosen sufficiently large. Therefore for c small, but positive the sublevel set $\{x : H^\phi(x) \leq c\}$ has a bounded component containing the origin.*

Proof: Introduce the *scaling* $z_1 = x_1$, $z_2^{2p-1} = x_2$. Then $\tilde{H}(z_1, z_2) = H(x_1, x_2) = z_1^{2p}\tilde{Q}(z_1) + z_2^{2p-1} z_1 \tilde{R}(z_1, z_2) + E z_2^{2p}$ (with $\tilde{Q}$ and $\tilde{R}$ analytic). Clearly, $\tilde{H}(0, z_2) > 0$ for $z_2 \neq 0$. Consider $\tilde{H}$ along the rays $z_2 = m z_1$: $\tilde{H}(z_1, m z_1) = (\tilde{Q}(0) + E m^{2p} + \tilde{R}(0,0) m^{2p-1}) z_1^{2p} + o(z_1^{2p})$. Upon choosing E sufficiently large the coefficient of z_1^{2p} is positive for all $m \in \mathbf{R}$ (using $\tilde{Q}(0) > 0$ and that $\tilde{R}$ is analytic) and therefore $\tilde{H}$, and thus H are positive in a deleted neighbourhood of the origin.

To obtain the precise critical value of the parameter E consider the scalar function $\psi(m) = \tilde{Q}(0) + E m^{2p} + \tilde{R}(0,0) m^{2p-1}$ which has an absolute minimum at $m_0 = (1 - 2p)\tilde{R}(0,0) \,/\, (2pE)$. Thus $\psi(m) > 0$ for all $m \in \mathbf{R}$, if and only if $\psi(m_0) > 0$, or in other words, iff

$$E > E_{\text{crit}} = \frac{2p-1}{2p} \left(\frac{\tilde{R}^{2p}(0,0)}{2p\,\tilde{Q}(0)} \right)^{1/(2p-1)} \tag{7}$$

3 Construction of the feedback law and an example

To construct a (locally) asymptotically stabilizing continuous feedback law for the general (small-time) locally controllable system (1) in the plane, the algorithm is as follows:

Algorithm: 1. Choose local coordinates such that the controlled field is $g(x) = \frac{\partial}{\partial x_1}$.
2. If necessary introduce the preliminary feedback $u = \mu(x) + \tilde{u}$ with $\mu(x) = -f_1(x)$, so that the system now takes the form $\dot{x}_1 = \tilde{u},\ \dot{x}_2 = f_2(x)$.
3. Determine the smallest integer ν such that $q_0 = \frac{\partial^\nu f_2}{\partial x_1^\nu}(0,0) \neq 0$. If the system is STLC then ν is odd, i.e. $\nu = 2p-1$ for a positive integer p. Set $\alpha = 1/(2p-1)$. If $q_0 > 0$ then change coordinates $x_2^{\text{new}} = -x_2^{\text{old}}$.
4. Form the function $H_E(x_1, x_2) = -\int f_2(x)dx_1 + Ex_2^{\alpha+1}$ where the constant of integration is chosen such that $H(0, x_2) = Ex_2^{\alpha+1}$ for all $x_2 \in \mathbf{R}$ and E is a sufficently large positive constant. See the previous section for the explicit critical value of E.
5. Let $u_0(x) = \frac{\partial H}{\partial x_2}$ and finally define the feedback law $\tilde{u}(x) = u_0(x) + Kf_2(x)$ (or $u(x) = \mu(x) + u_0(x) + Kf_2(x)$) where the *gain-parameter* K is any positive constant.

Theorem 3.1 *If the system (1) in the plane is small-time locally controllable then the feedback law constructed in above algorithm asymptotically stabilizes the system at the origin.*

Proof: We show that the so-defined locally Hölder continuous feedback law meets the transversality condition required in Proposition 2.2 for uniqueness of solutions of the closed-loop system (away from the origin), and that the function H may serve as a generalized Ljapunov function in a neighbourhood of the origin enabling us to conclude asymptotic stability.

First, note that (with the notation from the previous section) the only part of the feedback law that is not analytic is the term $\phi'(x_2) = (1+\alpha)Ex_2^\alpha$. However this term is locally Lipschitz everywhere off the x_1-axis and still is locally Hölder continuous with exponent $\alpha = 1/(2p-1)$ on the set $S = \{(x_1, 0) \ : \ x_1 \neq 0\}$. However, from the controllability hypothesis we have that $f_2(x)$ does not vanish on S, i.e. the transversality condition required in Proposition 2.2 is met by the constructed feedback; and hence through each point different from the origin there exists a unique solution.

In the previous section we established that the function H is positive on a deleted neighbourhood of the origin (provided E is chosen sufficiently large) and vanishes at $x = 0$. One easily sees that the origin is an isolated critical point of the function H, or in other words, that $x^0 = 0$ is an isolated point in the intersection of the zero sets of $u_0(x) = \partial H/\partial x_2$ and $f_2(x) = -\partial H/\partial x_1$. (This can most easily be seen after introducing a scaling like in the previous section from the fact that the function $\tilde{H}$ is analytic.) Therefore, $x^0 = 0$ is an isolated stationary point. Along trajectories of the closed loop system we compute

$$\begin{aligned}\tfrac{d}{dt}H(x_1,x_2) &= \tfrac{\partial H}{\partial x_1}\dot{x}_1 + \tfrac{\partial H}{\partial x_2}\dot{x}_2 \\ &= -f_2(x)(u_0(x)+Kf_2(x)) + u_0(x)f_2(x) \\ &= -Kf_2^2(x) \leq 0.\end{aligned} \tag{8}$$

We still have to verify that the intersection of the set $\{x : f_2(x) = 0\}$ with a sufficiently small neighbourhood of the origin does not contain any nontrivial half-trajectory: Let $\bar{x} \neq 0$ be such that $f_2(\bar{x}) = 0$ (i.e. $\bar{x}_2 \neq 0$ and hence f_2 has partial derivatives of abitrary order at $\bar{x}$), and assume that $f_2 \equiv 0$ and thus $\frac{d^j f_2}{dt^j} \equiv 0$ (for all $j = 0, 1, 2 \ldots$) along the solution curve through $\bar{x}$. From $\frac{df_2}{dt} = \frac{\partial f_2}{\partial x_1}(u_0 + Kf_2) + \frac{\partial f_2}{\partial x_2} f_2$ we conclude that then $\frac{\partial f_2}{\partial x_1} \equiv 0$ along this curve; and inductively, successive differentiations yield $\frac{\partial^j f_2}{\partial x_1^j} \equiv 0$ for all $j = 0, 1, \ldots$ along this curve, which contradicts the controllability hypothesis, since $\frac{\partial^\nu f_2}{\partial x_1^\nu}$ is nonzero at $x = 0$ and thus is nonzero in a neighbourhood.

We may remark here that these last calculations are very similar to those carried out in the proof of theorem 1 in Lee/Arapostathis [9], where also many successive differentiations are required. As an illustrating example we may look at the system $\dot{x}_1 = u, \dot{x}_2 = -x_1^m$, m an odd integer. Then $H(x) = x_1^{m+1}/(m+1) + Ex_2^{1+\alpha}$, $u_0(x) = ex_2^\alpha$ (here we may take $\alpha = 1/m$ or e.g. $\alpha = 1$), and $u(x) = u_0(x) + Kf_2(x)$ as above. Now if $\bar{x} = (0, \bar{x}_2)$ with $\bar{x}_2 \neq 0$, then at this point f, its first $(m-1)$ derivatives, and the first m derivatives of H vanish, but nonetheless H is strictly decreasing along each nontrivial trajectory. This phenomenon may be avoided by considering a different feedback law like $u(x) = u_0(x) + K\eta(x)$, where $\eta(x)$ is e.g. a reasonably nice function having the same sign as f_2 everywhere, but only vanishing to the first order where f_2 vanishes to a higher order; in this example we may take $\eta(x) = -x_1$, but in general η might be more difficult to find, or pose new problems, e.g., w.r.t. uniqueness of solutions; compare also the discussion at the end of the example in the following section.

Example 3.1 *We construct the locally Hölder continuous stabilizing feedback law for the following system (see also [1]):*

$$\begin{cases} \dot{x_1} &= u \\ \dot{x_2} &= x_2 - x_1^3. \end{cases} \tag{9}$$

This system is already in standard form, so that we may skip steps 1 and 2. Evaluating the partial derivatives of $f_2(x)$ at $x = 0$ we find $\nu = 2p - 1 = 3$, hence $\alpha = 1/3$. Next form $H(x_1, x_2) = x_1^4/4 - x_1x_2 + Ex_2^{4/3}$, choosing E sufficiently large. (Here $\tilde{H}(z_1, mz_1) = z_1^4(\frac{1}{4} - 1\,m^3 + Em^4)$, hence $\tilde{Q} \equiv \frac{1}{4}$, $\tilde{R} \equiv 1$, and thus $E_{\text{crit}} = \frac{3}{4}$.) Finally, form the stabilizing feedback law $u(x) = (-x_1 + \frac{4}{3}Ex_2^{1/3}) + K(x_2 - x_1^3)$. where K is any positive constant determining the gain.

Note that there does not exist a C^1-stabilizing feedback law for system (9) because it fails to satisfy condition (i) from theorem 1.2 (the linearization is $\dot{x}_1 = u,\ \dot{x}_2 = x_2$). However, the standard nilpotent approximation $\dot{x}_1 = u,\ \dot{x}_2 = -x_1^3$ with dilation exponents 1 and 3 can easily be feedback stabilized.

In the feedback law the term $Kf_2(x)$ may be replaced by any other sufficiently nice function $K\eta$ that always has the same sign, e.g. one may take $K\eta = K(x_2^{1/3} - x_1)$. Combining this particular term with the u_0-part of the feedback we obtain the feedback law $\hat{u} = -(1+K)x_1 + (K + \frac{4}{3}E)x_2^{1/3}$, which is of the simple form $\hat{u} = -ax_1 + bx_2^{1/3}$. Simple analysis shows that this feedback law asymptotically stabilizes the system at the origin provided $b > a > 1$.

Remark: The geometric meaning of the parameter E is a generalized excentricity of the level sets of the function H. In particular, for small values of E the level sets $H = c$ (for small positive values of c) no longer have a bounded component: indeed, H then has both positive and negative values in every neighbourhood of the origin. The parameter K is similar to the gain in the linear case and essentially describes how fast the solutions spiral in towards the origin.

4 Concluding remarks and acknowledgements

An immediate generalization of above explicit construction of asymptotically stabilizing locally Hölder continuous feedback laws to higher dimension can not be expected, since the explicit nature of the given algorithm depends in a strong way on the possibility of firstly choosing a feedback law that makes the closed loop system Hamiltonian and secondly to be able to tune the feedback law so that the solutions of the Hamiltonian system stay bounded. Unless one considers very special systems, the only hope for obtaining a similar setting are $m-$input systems evolving on $2m-$dimensional space; but even here an analogue of above construction remains still complicated.

The other main idea of this paper, the employment of sublinear, here locally Hölder continuous feedback laws for the asymptotic stabilization of (small-time) locally controllable systems, that can not be asymptotically stabilized with C^1-feedback laws, is expected to be applicable to a large class of systems, and is certainly not restricted to systems evolving on two-dimensional space. In particular we conjecture that in general the *largest possible* Hölder exponent of an asymptotically stabilizing feedback law is inversely related to the length of the iterated Lie brackets in f and g that *give* (small-time) local controllability, compare [8, 15]. This, and the investigation of the circumstances under which such a feedback law also asymptotically stabilizes perturbations of the original system are subject of continuing research.

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