

# An Angular Open Mapping Theorem

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We give an angular open mapping theorem applicable to local approximating cones of the attainable set of an affine control system. The proof is constructive and indicates how to implement a discrete time stabilizing feedback.

## 1 Introduction

The purpose of this paper is to provide a modified open mapping theorem, which also applies to maps which are not open, but whose intersection with a convex cone is relatively open. We give a constructive proof, which for the special maps considered here may be seen as an alternative to the indirect proof of Frankowska's more general open mapping principle.

Open mapping theorems are of utmost importance in both the theories of optimality and local controllability, as only they allow one to use local approximating cones to conclude that a trajectory lies on the boundary (or in the interior) of the attainable set. In the theory of optimal control, e. g. the Maximum Principle (e.g. [4, 5, 6]), one typically uses a very restrictive definition of *tangent vectors to a set* which leads to small approximating cones. This is closely related to the fact that in general the attainable set may have inward corners (even inward cusps). This makes it impossible to work in that case with the most general – and most intuitive – notion of tangent vectors to a set and simultaneously have them form a convex cone, which is also an approximating cone to the attainable set.

However, to obtain strong sufficient conditions for (small-time) local controllability (STLC) about an equilibrium point one wants to employ the most general notion of tangent vectors to a set. (For example, one does not want to be restricted to tangent vectors generated by families of Pontryagin control variations, or those employed in the High Order Maximal Principle [4], but instead be able to also consider tangent vectors generated by, e.g., discretely parametrized families of control variations with an increasing number of switchings like those employed in [3].

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Of course, by enlarging the set of tangent vectors, one needs a stronger open mapping theorem, e.g., like the one provided in [1, 2]. For the special case of an affine Lipschitz control system we give a direct proof of a similar result, which also allows to draw the "expected" conclusions in the case that the approximating cone is not the full space. Also, as a side result we obtain information about (directional) Hölder continuity of the minimum time value function.

## 2 Preliminaries

We consider affine control systems on  $\mathbf{R}^n$  of the form

$$\dot{x} = f(x) + \sum_{j=1}^{\kappa} u_j(t)g^j(x), \quad x(0) = 0, \quad u(\cdot) \in \mathcal{U} \quad (1)$$

where the vectorfields  $f$  and  $g^j$  are locally Lipschitz near zero,  $f(0) = 0$ , and the controls  $u^j$  are measurable with values in the compact set  $\mathcal{U} \subseteq \mathbf{R}^{\kappa}$ , where  $0 \in \text{int}\mathcal{U}$ . The solution to (1) (at time  $t \geq 0$ ) corresponding to control  $u$  is denoted by  $x(t, u) = x(t, u)(0)$ . (Sometimes we will also use  $x(t, u)(x_0)$  to denote solution to (1) with initial condition  $x(0) = x_0$ .) The attainable set at time  $t \geq 0$  is  $\mathcal{A}(t) = \{x(t, u) : u(\cdot) \in \mathcal{U}\}$ . The system is small-time locally controllable (STLC) about the rest solution  $x \equiv 0$  if  $0 \in \text{int}\mathcal{A}(t)$  for all  $t > 0$ . Otherwise  $x \equiv 0$  lies on the boundary of the attainable funnel  $\bigcup_{t>0}\{t\} \times \mathcal{A}(t) \subseteq \mathbf{R}^{n+1}$  for small  $t \geq 0$  and thus may be considered optimal.

In the proof of the theorem, and also in the following lemma, we will use Gronwall's lemma several times in the following form: If  $f, g^1, \dots, g^{\kappa}$  have the Lipschitz constant  $L$  on  $\mathcal{A}(2T)$  in the sense that for all  $y_1, y_2 \in \mathcal{A}(2T)$ ,  $u \in \mathcal{U}$

$$\left| f(y_1) - f(y_2) + \sum_{j=1}^{\kappa} u_j(g^j(y_1) - g^j(y_2)) \right| < L|y_1 - y_2| \quad (2)$$

and  $p \in \mathcal{A}(T)$  then  $|x(t, u)(p) - x(t, u)(0) - p| \leq |p|te^{Lt}$  for all  $0 \leq t \leq T$  and  $u(\cdot) \in \mathcal{U}$ . In particular, for any  $\delta > 0$  we may choose  $T = T_{\delta} > 0$  such that  $T_{\delta}e^{LT_{\delta}} \leq \delta$  and thus  $|x(t, u)(p) - x(t, u)(0) - p| \leq \delta T_{\delta}$  for all  $0 \leq t \leq T_{\delta}$ ,  $p \in \mathcal{A}(T_{\delta})$ .

Among the various possible notions of tangent vectors to the attainable set, we use the following:

**Definition 2.1** *A vector  $v \in \mathbf{R}^n$  is a  $m$ -th order tangent vector to the attainable sets (of (1)) at zero if for all  $t \geq 0$  there are points  $q(t) \in \mathcal{A}(t)$  of the form  $q(t) = vt^m + o(t^m)$ . (Here  $o(t^m)$  stands for terms such that  $\lim_{t \rightarrow 0} o(t^m)/t^m = 0$ .)*

The set of all  $m$ -th order tangent vectors (to  $\mathcal{A}(t)$  at zero) is denoted by  $\mathcal{K}^m$ , while  $\overline{\mathcal{K}^m} = \bigcup_{\lambda>0} \lambda\mathcal{K}^m$  is the set of *tangent rays* to  $\mathcal{A}(t)$  at zero. With this notion we have:

**Lemma 2.1** (a) *If  $\lambda^m \in [0, 1]$ , then  $\lambda^m\mathcal{K}^m \subseteq \mathcal{K}^m$ .*

(b) If  $m \leq l$  then  $\mathcal{K}^m \subseteq \mathcal{K}^l$ .

(c) If  $v_1, v_2 \in \mathcal{K}^m$  and  $\lambda^m \in [0, 1]$  then  $\lambda^m v_1 + (1 - \lambda)^m v_2 \in \mathcal{K}^m$ .

Thus the sets  $\mathcal{K}^m$  form an increasing sequence of truncated convex cones. To keep this paper self-contained we give the short proof here (alternatively see e.g. [2]).

**Proof.** (a) Let  $\lambda^m \in [0, 1]$ ,  $v \in \mathcal{K}^m$  and choose a one-parameter family  $\{u_s\}_{s \geq 0}$  of control variations  $u_s : [0, s] \rightarrow \mathcal{U}$  such that  $x(s, u_s) = v s^m + o(s^m)$ ,  $s \geq 0$ . Define the controls  $\tilde{u}_s : [0, s] \rightarrow \mathcal{U}$  by

$$\tilde{u}_s(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq (1 - \lambda)s \\ u_s(t - (1 - \lambda)s) & \text{if } (1 - \lambda)s < t \leq s \end{cases} \quad (3)$$

and clearly  $x(s, \tilde{u}_s) = x(\lambda s, u_{\lambda s}) = (\lambda^m v) s^m + \lambda^m o(s^m)$ .

(b) Start with  $\lambda, v$  and  $\{u_s\}_{s \geq 0}$  as above. Also, let  $l \geq m$ . Clearly, we only have to consider  $s \leq 1$  and thus may define, using  $\sigma = s^{m/l}$ ,

$$\bar{u}_s(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq s - \sigma \\ u_s(t - \sigma) & \text{if } \sigma - s < t \leq s \end{cases} \quad (4)$$

and clearly  $x(s, \bar{u}_s) = x(\sigma, u_\sigma) = v \sigma^l + o(\sigma^l) = v s^m + o(s^m)$ .

(c) Let  $\{u_s^{(1)}\}_{s \geq 0}$  and  $\{u_s^{(2)}\}_{s \geq 0}$  be families of control variations generating the  $m$ -th order tangent vectors  $v^1$  and  $v^2$ , respectively. For  $\lambda \in [0, 1]$  define the family  $\{u_s^\lambda\}_{s \geq 0} : [0, s] \rightarrow \mathcal{U}$  by

$$u_s^\lambda(t) = \begin{cases} u_{\lambda s}^{(1)}(t) & \text{if } 0 \leq t \leq \lambda s \\ u_{1-\lambda s}^{(2)}(t - \lambda s) & \text{if } \lambda s < t \leq s. \end{cases} \quad (5)$$

The conclusion  $x(s, u_s^\lambda) = (\lambda^m v^1 + (1 - \lambda)^m v^2) s^m + o(s^m)$  is an immediate consequence of Gronwall's lemma. Note that the properties (a) and (b) of the truncated cones  $\mathcal{K}^m$  strongly depend on  $x \equiv 0$  being an equilibrium solution. Also, even if  $s \rightarrow x(s, u_s^{(1)})$  and  $s \rightarrow x(s, u_s^{(2)})$  are smooth curves, in general  $s \rightarrow x(s, u_s^\lambda)$  will not be smooth, usually not even a curve, as is illustrated in the following.

### Example 2.1

Let  $f(x) = 12x_1^2 \frac{\partial}{\partial x_2} = (0, 12x_1^2)^T$  and  $g(x) = \frac{\partial}{\partial x_1} = (1, 0)^T$ . Define

$$u_s^{(1)}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq s - s^3 \\ 1 & \text{if } s - s^3 < t \leq s \end{cases} \quad (6)$$

Let  $I' \subseteq \mathbf{R}_0^+$  be an arbitrary set of nonnegative numbers and  $I'' = \mathbf{R}_0^+ \setminus I'$  its complement. If  $s \in I'$  then define

$$u_s^{(2)}(t) = \begin{cases} +1 & \text{if } 0 \leq t \leq \frac{s}{2} \\ -1 & \text{if } \frac{s}{2} < t \leq s \end{cases} \quad (7)$$

and if  $s \in I''$  then define

$$u_s^{(2)}(t) = \begin{cases} -1 & \text{if } 0 \leq t \leq \frac{s}{2} \\ +1 & \text{if } \frac{s}{2} < t \leq s \end{cases} \quad (8)$$

Then the terminal points  $x(s, u_s^{(1)}) = (s^3, 4s^9)$  and  $x(s, u_s^{(2)}) = (0, s^3)$  lie on smooth curves and the families  $\{u_s^{(1)}\}_{s \geq 0}$  and  $\{u_s^{(2)}\}_{s \geq 0}$  generate the third order tangent vectors  $\frac{\partial}{\partial x_1}$  and  $\frac{\partial}{\partial x_2}$ , respectively. However, if for  $\lambda \in (0, 1)$  we define

$$u_s^\lambda(t) = \begin{cases} u_s^{(1)}(t) & \text{if } 0 \leq t \leq \lambda s \\ u_s^{(2)}(t) & \text{if } \lambda s < t \leq s, \end{cases} \quad (9)$$

then

$$x(s, u_s^\lambda) = \begin{cases} (\lambda^3 s^3, ((1-\lambda)s + 2(\lambda s)^3)^3 - 4(\lambda s)^9) & \text{if } s \in I' \\ (\lambda^3 s^3, ((1-\lambda)s - 2(\lambda s)^3)^3 + 12(\lambda s)^9) & \text{if } s \in I'' \end{cases} \quad (10)$$

and in general  $s \rightarrow x(s, u_s^\lambda)$  will no longer be a smooth curve, e. g. choose both  $I'$  and  $I''$  dense in  $\mathbf{R}_0^+$ . However, the families  $\{u_s^\lambda\}_{s \geq 0}$  still generate the third order tangent vectors  $\lambda^3 \frac{\partial}{\partial x_1} + (1-\lambda)^3 \frac{\partial}{\partial x_2}$ , in the sense as in definition 2.1.

### 3 Motivation and statement of the theorem

Consider the single input system

$$\begin{cases} \dot{x}_1 = u & |u(\cdot)| \leq 1 \\ \dot{x}_2 = x_1^2 & x(0) = 0. \end{cases} \quad (11)$$

The attainable set at time  $T \geq 0$  is

$$\mathcal{A}(t) = \{x \in \mathbf{R}^2 : |x_1| \leq t, \frac{1}{3}x_1^3 \leq x_2 \leq \frac{1}{12}(t+x_1)^3 - \frac{1}{3}x_1^3\} \quad (12)$$

while  $\overline{\mathcal{K}^3} = \mathbf{R} \times \mathbf{R}_0^+$ , the closed upper half plane. Thus no nontrivial truncation of  $\mathcal{K}^3$  lies inside the attainable set  $\mathcal{A}(t)$  at any time  $t > 0$ . However, for every closed convex cone  $\overline{\mathcal{K}'}$  (with vertex at zero) such that  $\overline{\mathcal{K}'} \setminus \{0\} \subseteq \text{int} \overline{\mathcal{K}^3}$  there are positive constants  $C, T > 0$  such that  $\overline{\mathcal{K}'} \cap B(0, Ct^3) \subseteq \mathcal{A}(t)$  for all  $0 \leq t \leq T$ , see also figure 1.



Figure 1: Attainable set and approximating cones

This picture already contains the main idea, as we have the following

**Theorem 3.1** *If  $\overline{\mathcal{K}'}$  is a closed convex cone (with vertex  $0 \in \mathbf{R}^n$  such that  $\overline{\mathcal{K}'} \setminus \{0\} \subseteq \text{int}\overline{\mathcal{K}^m}$  for some  $m < \infty$ , then there are constants  $C > 0$ ,  $T > 0$  such that  $\overline{\mathcal{K}'} \cap B(0, Ct^m) \subseteq \mathcal{A}(t)$  for all  $0 \leq t \leq T$ .*

**Corollary 3.2** *If  $\overline{\mathcal{K}^m} = \mathbf{R}^n$  then there are constants  $C > 0$ ,  $T > 0$  such that  $B(0, Ct^m) \subseteq \mathcal{A}(t)$  for all  $0 \leq t \leq T$ .*

We conclude this section with the following remarks:

1. The theorem may also be thought of as the statement that the minimum-time map is locally Hölder continuous with exponent  $1/m$  in the direction of  $\overline{\mathcal{K}^m}$  (or in a full neighbourhood of zero, in the case of a full tangent cone as in the Corollary).
2. At this time we do not know if the following is true: “If the vectorfields  $f, g^1, \dots, g^k$  are real analytic and the system (1) is small-time locally controllable about zero, is then  $\overline{\text{cal}\mathcal{K}^m} = \mathbf{R}^n$  for some finite  $m$ ?” (Or alternatively, does small-time local controllability imply local Hölder continuity of the minimum-time map near zero?). In the  $C^\infty$ -case, the system may be small-time locally controllable, but the approximating cones  $\overline{\mathcal{K}^m}$ ,  $m > 0$  may all have empty interior as is clear from the system  $\dot{x}_1 = u$ ,  $\dot{x}_2 = \exp(-1/x_1^2)$ ,  $x(0) = (0, 0)$ ,  $|u(\cdot)| \leq 1$ . While generalizations of above theorem covering such systems may be possible, certain difficulties arise in the proof when composing controls which generate tangent vectors of “infinite order”.

## 4 Proof of the theorem

To facilitate the proof of the theorem we reduce it to the case of  $\overline{\mathcal{K}'}$  being an elementary simplicial cone  $\mathcal{C}(v^1, \dots, v^n) = \{\sum_{j=1}^n \lambda_j v^j : \lambda_j \geq 0\}$  generated by the  $n$  vectors  $v^1, \dots, v^n$  by means of the following

**Fact:** *If  $\overline{\mathcal{K}'}$  and  $\overline{\mathcal{K}}$  are closed convex cones (with vertex zero) such that  $\overline{\mathcal{K}'} \setminus \{0\} \subseteq \text{int}\overline{\mathcal{K}}$ , then there is a finite number of elementary simplicial cones  $\mathcal{C}^i(v^{i,1}, \dots, v^{i,n})$ ,  $i = 1, \dots, \rho$ , such that  $\mathcal{C}^i \setminus \{0\} \subseteq \text{int}\overline{\mathcal{K}}$  for each  $i = 1, \dots, \rho$  and  $\overline{\mathcal{K}'} \subseteq \cup_{i=1}^\rho \mathcal{C}^i$ .*

A straightforward way to verify this fact is by considering the open cover of the compact set  $\overline{\mathcal{K}'} \cap S^{n-1}$  by the interiors of all elementary simplicial cones which are contained in  $\overline{\mathcal{K}}$ , and then taking a finite subcover (here  $S^{n-1}$  is the unit  $(n-1)$ -sphere).

Next we may assume that the elementary simplicial cone  $\overline{\mathcal{K}'} = \mathcal{C}(v^1, \dots, v^n)$  has nonempty interior, and thus there is a linear change of coordinates such that (in the new coordinates)  $e^i = (0, 0, \dots, 0, 1, 0, \dots, 0) \in \overline{\mathcal{K}}$  and  $v^i = (\varepsilon, \varepsilon, \dots, 1 - (n-1)\varepsilon, \varepsilon, \dots, \varepsilon) = (1 - n\varepsilon)e^i + \sum_{j=1}^n \varepsilon e^j$ ,  $i = 1, \dots, n$  for some  $\varepsilon > 0$  which we may take sufficiently small so that it satisfies  $4n^2\varepsilon \leq 1$ .

Note that  $w = (w_1, \dots, w_n) \in \overline{\mathcal{K}'}$  if and only if  $|w_i| \geq \varepsilon|w| = \varepsilon \sum_{j=1}^n |e_j|$ ,  $i = 1, \dots, n$ . We work with the associated 1-norm on  $\mathbf{R}^n$  (w.r.t. the basis  $(e^1, \dots, e^n)$ ).

This, and the change of coordinates clearly may affect the constant  $C$  in the theorem (in  $B(0, Ct^2)$ ) by multiplication with a positive factor.

As outlined above, there is  $T_1 > 0$  such that for all  $0 < T \leq T_1$ ,  $p \in \mathcal{A}(T_1)$ , and  $u(\cdot)$  admissible we have  $|x(t, u)(p) - x(t, u)(0) - p| \leq \frac{\varepsilon}{4n}|p|$  as a consequence of Gronwall's inequality and because the vectorfields  $f, g^1, \dots, g^n$  are assumed to be Lipschitz near zero.

Fix families  $\{u_s^i\}_{s \geq 0}$  of admissible controls generating the  $m$ -th order tangent vectors  $e^i$ ,  $i = 1, \dots, n$ . Fix  $0 < T \leq T_1$  such that  $|x(t, u_t^i)(0) - t^m e^i| \leq \varepsilon^2 t^m$  for  $0 \leq t \leq T$  and  $i = 1, \dots, n$ .

Let  $C = (2nm)^{-m}$ . To show that  $\overline{\mathcal{K}'} \cap B(0, Ct^m) \subseteq \mathcal{A}(t)$  for all  $0 \leq t \leq T$  we will for any fixed  $z^\infty \in \overline{\mathcal{K}'} \cap B(0, Ct^m)$  construct a control  $\bar{v}^\infty$  (as a  $L^1$ -limit) such that  $x(t, \bar{v}^\infty)(0) = z^\infty$ .

Fix  $z^\infty \in \overline{\mathcal{K}'} \cap B(0, CT^m)$ , and define  $\tau_\infty = 2nm \cdot |z^\infty|^{1/m}$ .

Before we begin the construction, we need to introduce further notation: If  $u$  defined on the time interval  $[0, \tau]$  with  $\tau \leq \tau_\infty$  is an admissible control, then define the control  $\bar{u}$  on the time interval  $[0, \tau_\infty]$  by

$$\bar{u}(t) = \begin{cases} 0 & \text{if } 0 \leq t < \tau_\infty - \tau \\ u(t - (\tau_\infty - \tau)) & \text{if } \tau_\infty - \tau \leq t \leq \tau_\infty \end{cases} \quad (13)$$

Then clearly  $x(\tau_\infty, \bar{u})(0) = x(\tau, u)(0)$ , using  $f(0) = 0$ .

We let  $z^0 = 0$ ,  $\tau_0 = 0$ ,  $\bar{v}^0 \equiv 0$  and define a sequence of admissible controls  $v^i : [0, \tau_i] \rightarrow \mathcal{U}$  such that  $z^i \stackrel{\text{def}}{=} x(\tau_\infty, \bar{v}^i)(0) \rightarrow z^\infty$ . Since  $z^i \in \mathcal{A}(\tau_\infty)$  for each  $i$ , and  $\mathcal{A}(\tau_\infty)$  is closed,  $z^\infty \in \mathcal{A}(\tau_\infty)$  follows. Moreover,  $\bar{v}^i \xrightarrow{L^1} \bar{v}^\infty$  and  $x(\tau_\infty, \bar{v}^\infty)(0) = z^\infty$ .

Suppose that  $z^i$ ,  $\tau_i$  and  $v^i : [0, \tau_i] \rightarrow \mathcal{U}$  have been defined. Choose  $\mu(i+1) \in \{1, \dots, n\}$  such that  $(z^\infty - z^i)_{\mu(i+1)} \geq (z^\infty - z^i)_l$  for  $l = 1, \dots, n$ .

Let  $t_{i+1} = \left( (1 - n\varepsilon)(z^\infty - z^i)_{\mu(i+1)} \right)^{1/m}$  and  $\tau_{i+1} = \tau_i + t_{i+1}$  ( $= \sum_{j=0}^{i+1} t_j$ ). Define

$$v^{i+1}(s) = \begin{cases} u_{t_{i+1}}^{\mu(i+1)}(s) & \text{if } 0 \leq s \leq t_{i+1} \\ v^i(s - t_{i+1}) & \text{if } t_{i+1} \leq s \leq \tau_{i+1} \end{cases} \quad (14)$$

Let  $y^{i+1} = x(t_{i+1}, u_{t_{i+1}}^{\mu(i+1)})(0)$  and  $z^{i+1} = x(\tau_{i+1}, v^{i+1})(0) = x(\tau_i, v^i)(y^{i+1})$ .

To verify that  $|z^\infty - z^i| \rightarrow 0$  as  $i \rightarrow \infty$  first observe that

$$|y^{i+1} - t_{i+1}^m e^{\mu(i+1)}| \leq \varepsilon^2 t_{i+1}^m \quad (15)$$

and thus

$$|y^{i+1}| \leq (1 + \varepsilon^2 t_{i+1}^m) |z^\infty - z^i| \quad (16)$$

Using Gronwall's inequality and (16) as an estimate for  $|y^{i+1} - 0|$  we have

$$|z^{i+1} - (z^i + y^{i+1})| \leq \frac{\varepsilon}{4n} |y^{i+1}|. \quad (17)$$

By choice of  $\mu(i+1)$  we have

$$\begin{aligned} |(1-n\varepsilon)(z^\infty - z^i) - t_{i+1}^m e^{\mu(i+1)}| &= (1-n\varepsilon) \sum_{l \neq \mu(i+1)} (z^\infty - z^i)_l \\ &\leq (1-n\varepsilon) \frac{n-1}{n} |z^\infty - z^i|. \end{aligned} \quad (18)$$

Combining the estimates (15), (17) and (18) we obtain

$$\begin{aligned} |z^\infty - z^{i+1}| &\leq n\varepsilon |z^\infty - z^i| + |(1-n\varepsilon)(z^\infty - z^i) - t_{i+1}^m e^{\mu(i+1)}| \\ &\quad + |t_{i+1}^m e^{\mu(i+1)} - y^{i+1}| + |z^{i+1} - (z^i + y^{i+1})| \\ &\leq (n\varepsilon + (1-n\varepsilon) \frac{n-1}{n} + \varepsilon^2 + \frac{\varepsilon}{4n}) |z^\infty - z^i| \\ &\leq \frac{2n-1}{2n} \text{ (by choice of } \varepsilon \leq \frac{1}{4n^2} \text{)}. \end{aligned}$$

Thus, inductively

$$|z^\infty - z^i| \leq \left( \frac{2n-1}{2n} \right)^{i+1} \longrightarrow 0 \text{ as } i \longrightarrow \infty \quad (19)$$

Next observe that  $t_{i+1} \leq |z^\infty - z^i|^{1/m} \leq \left( \left( \frac{2n-1}{2n} \right)^i |z^\infty| \right)^{1/m}$  and therefore

$$\begin{aligned} \tau_i &= \sum_{j=0}^i t_j \leq |z^\infty|^{1/m} \sum_{j=0}^i \left( \frac{2n-1}{2n} \right)^{j/m} < |z^\infty|^{1/m} \frac{1}{1 - \left(1 - \frac{1}{2n}\right)^{1/m}} \\ &\leq 2nm |z^\infty|^{1/m} \leq \tau_\infty. \end{aligned}$$

(Here we used that  $\left(1 - \frac{1}{2n}\right)^{1/m} \leq 1 - \frac{1}{2nm}$ .)

In this last calculation it becomes apparent how the finite order  $m$  of the tangent vectors allows one to do infinitely many successive ‘‘corrections’’ without needing an unbounded amount of time.

Finally we have to verify that  $(z^\infty - z^{i+1}) \in \overline{\mathcal{K}'}$  for all  $i$ . (This is necessary since we do not allow to follow trajectories backwards in time, i.e. we require  $t_{i+1} \geq 0$  for each  $i$ .)

By hypothesis  $z^\infty - z^0 = z^\infty \in \overline{\mathcal{K}'}$ . Now suppose  $(z^\infty - z^i) \in \overline{\mathcal{K}'}$ , i. e.,  $(z^\infty - z^i)_l \geq \varepsilon |z^\infty - z^i|$ ,  $l = 1, \dots, n$ . Also,  $(z^\infty - z^i)_{\mu(i+1)} \geq \frac{1}{n} |z^\infty - z^i|$  by choice of  $\mu(i+1)$ . Using these together with the inequalities (15), (17) and (18) componentwise, we find:

$$(z^\infty - z^{i+1})_{\mu(i+1)} - n\varepsilon (z^\infty - z^i)_{\mu(i+1)} \leq \left( \varepsilon^2 + \frac{\varepsilon}{4n} \right) |z^\infty - z^i| \quad (20)$$

and thus

$$\begin{aligned} \frac{(z^\infty - z^{i+1})_{\mu(i+1)}}{|z^\infty - z^{i+1}|} &\geq \frac{n\varepsilon (z^\infty - z^i)_{\mu(i+1)} - \left( \varepsilon^2 + \frac{\varepsilon}{4n} \right) |z^\infty - z^i|}{\frac{2n-1}{2n} |z^\infty - z^i|} \\ &\geq \frac{2n}{2n-1} \left( \frac{1}{n} n\varepsilon - \varepsilon^2 - \frac{\varepsilon}{4n} \right) \\ &\geq \varepsilon \cdot \frac{4n-1-\varepsilon}{2(2n-1)} \geq \varepsilon \text{ by choice of } \varepsilon \end{aligned}$$

Similarly for  $l \neq \mu(i+1)$

$$(z^\infty - z^{i+1})_l = (z^\infty - z^i)_l + (y^{i+1} + z^i - z^{i+1})_l - y_l^{i+1} \quad (21)$$

and thus

$$\begin{aligned} \frac{(z^\infty - z^{i+1})_l}{|z^\infty - z^{i+1}|} &\geq \frac{(z^\infty - z^{i+1})_l - (\varepsilon^2 + \frac{\varepsilon}{4n})|z^\infty - z^{i+1}|}{\frac{2n-1}{2n}|z^\infty - z^{i+1}|} \\ &\geq \frac{2n}{2n-1}(\varepsilon - \varepsilon^2 - \frac{\varepsilon}{4n}) \geq \varepsilon. \end{aligned}$$

Here it becomes apparent, that to make the inductive step work, one needs that  $\overline{\mathcal{K}'} \setminus \{0\}$  is properly contained in the interior of  $\overline{\mathcal{K}}$ , and that one also has to “aim” at a point short of  $(z^\infty - z^i)$  in order to avoid going too far (because of the several error terms involved), so that further corrections are still possible.

Even though the corollary follows immediately from the theorem, e. g., by applying the theorem to the  $2^n$  elementary simplicial cones  $\mathcal{C}(\pm e^1, \dots, \pm e^n)$  for all possible combinations of + and - signs, a considerably shorter direct proof can be obtained upon observing that in this case one can at each step “shoot” directly for  $z^\infty - z^i$ , since now corrections in all directions are possible, i.e., one has no longer to worry about “over-shooting”.

Specifically, without introducing the cone  $\overline{\mathcal{K}'}$  and  $\varepsilon$ , via a linear change of coordinates (e.g. pure scaling) one may assume that  $\overline{\mathcal{K}} \subseteq B(0, 1)$ . Choose families  $\{u_s^{\pm i}\}_{s \geq 0}$  of controls generating the tangent vectors  $\pm e^n$ ,  $i = 1, \dots, n$ . Now choose  $t_{i+1} = |(z^\infty - z^i)_{\mu(i+1)}|$  and use as the next control  $u_{t_{i+1}}^{+\mu(i+1)}$  if  $(z^\infty - z^i)_{\mu(i+1)} > 0$  and  $u_{t_{i+1}}^{-\mu(i+1)}$  otherwise. This avoids the previous delicate choice of  $t_{i+1}$  and one may expect to arrive at the point  $z^\infty$  in even considerably smaller time, which essentially allows one to choose a larger value of the constant  $C$ .

## 5 Example and application to stabilization

As an example how the theorem may be used to decide that a system is small-time locally controllable, or that the attainable set at small times contains a truncated cone, we consider the following system (which has been discussed in detail in [3]).

$$\begin{cases} \dot{x}_1 = u & x(0) = 0 \\ \dot{x}_2 = x_1 & |u(\cdot)| \leq 1 \\ \dot{x}_3 = x_1^3 \\ \dot{x}_4 = x_3^2 - x_2^m & (m = 7 \text{ or } m = 8) \end{cases} \quad (22)$$

All the presently known general theorems for sufficient conditions for small-time local controllability fail to provide any definite conclusion for this system. However, the tangent vectors  $\pm c_i \frac{\partial}{\partial x_i}$ ,  $i = 1, 2, 3$ , and  $+c_4 \frac{\partial}{\partial x_4}$  (with suitable positive constants  $c_i$ ) of respective orders 1, 2, 4 and 9 are easily generated.

Also, in the case of  $m = 7$  one may generate  $-c_4 \frac{\partial}{\partial x_4}$  as a tangent vector (of order 57) in the following way: Define  $u^1 : [0, 3 + 2\sqrt{2}] \rightarrow [-1, 1]$  by

$$u^1(t) = \begin{cases} 1 & \text{if } t \in [0, \sqrt{2}) \cup [2 + 2\sqrt{2}, 3 + 2\sqrt{2}] \\ 0 & \text{if } t \in [1 + 2\sqrt{2}, 2 + 2\sqrt{2}) \\ -1 & \text{if } t \in [\sqrt{2}, 1 + 2\sqrt{2}) \end{cases} \quad (23)$$

Let  $u^{-1}$  be the time-reversed control defined by  $u^{-1}(t) = u^1(T - t)$  (where  $T = 3 + 2\sqrt{2}$ ), and define inductively via concatenation  $\tilde{u}^{(1)} = u^{-1} * u^1$  and  $\tilde{u}^{(k+1)} = u^{-1} * \tilde{u}^{(k)} * u^1 : [0, 2kT] \rightarrow [-1, 1]$ . Finally, we use from [3] that there is a positive constant  $M$  such that if one chooses  $k(s)$  (for  $s > 0$ ) to be the smallest integer larger than  $Ms^{-6}$ , then  $-c_4 \frac{\partial}{\partial x_4}$  is generated as a 57-th order tangent vector by the family of controls  $u_s : [0, s] \rightarrow [-1, 1]$ ,  $s \geq 0$  defined by  $u_s(2k(s)Tt) = \tilde{u}^{(k(s))}(t)$ . Thus, in the case of  $m = 7$ , we have  $\overline{\mathcal{K}}^{57} = \mathbf{R}^4$  and therefore may conclude that there are constants  $C, T > 0$  such that  $\mathcal{A}(t) \supseteq B(0, Ct^{57})$  for all  $t \in [0, T]$ , and in particular the system is small-time locally controllable.

In [3] it is shown that if  $m = 8$ , then  $x_4(t, u) \geq 0$  for all  $t > 0$  sufficiently small and all admissible controls  $u$ . Consequently, in this case one obtains  $\overline{\mathcal{K}}^9 = \{y = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4 : y_4 \geq 4\}$ . Letting  $\overline{\mathcal{K}}'_\delta = \{y \in \mathbf{R}^4 : y_4 \geq \delta(|y_1| + |y_2| + |y_3|)\}$ , for each  $\delta > 0$  there are  $C_\delta, T_\delta > 0$  such that  $\mathcal{A}(t) \supseteq B(0, C_\delta t^9) \cap \overline{\mathcal{K}}'_\delta$  for  $t \in [0, T_\delta]$ . While the minimum-time function clearly is discontinuous at the origin, we may say it is *Hölder continuous with exponent 1/9 in the direction of  $\overline{\mathcal{K}}^9$  at zero*.

The constructive proof we have given here for the theorem can almost immediately be employed to synthesize a special form of stabilizing feedback for the time reversed system

$$\dot{x} = -f(x) - \sum_{j=1}^{\kappa} u_j g^j(x), \quad x(0) = z^\infty, \quad u \in \mathcal{U} \quad (24)$$

in the following sense: Suppose one knows a finite number of one-parameter families of controls  $\{u_s^i\}_{s \geq 0}$ ,  $i = 1, \dots, r$ , generating the tangent vectors  $v^i \in \mathcal{K}^m$  for the system (1), and as before  $\overline{\mathcal{K}}'$  is a closed convex cone with  $\overline{\mathcal{K}}' \setminus \{0\} \subseteq \text{int} \overline{\mathcal{K}}^m$ . Then for each initial state  $z^\infty \in \overline{\mathcal{K}}'$  sufficiently close to zero one may construct a control  $\tilde{v}^\infty$  steering the system (24) to zero in the finite time  $\tau_\infty$ ; the control  $\tilde{v}^\infty$  is essentially the time reversed control of  $\overline{v}^\infty$  as constructed in the proof of the theorem.

Note that this is not a feedback-law in the usual sense (i.e.  $u = u(x)$ ), in particular it is not memoryless. However, an advantage of this feedback for implementations is that one only has to know a finite number of one-parameter families of controls (rather than one n-parameter family). Furthermore, only at discrete times does one have to determine the state, and then to decide which control to use next (and for how long). The time-intervals in between consecutive decisions naturally become shorter, the closer the system is to the equilibrium.

Finally, from this point of view it also is natural to first take a “rough shot” at the origin, and then make successively finer corrections thereafter – while in the

proof of the theorem the order in which the successive corrections are composed seemed unnatural at first; which now clearly can be attributed to the reversed direction of the time. Moreover, this form of stabilization is also self-correcting (as long as the errors are not too large) and even allows one to consider systems where the rest point for some reason may be reached only from certain directions forming a cone  $\overline{\mathcal{K}}$  properly contained in  $\mathbf{R}^n$ , e.g., the soft landing problem.

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