# On the problem whether controllability is finitely determined

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***Abstract*— Controllability of finite dimensional linear systems can be decided via a finite number of matrix operations, with an a-priori known bound on this number of operations. For both polynomial and general nonlinear analytic systems that are affine in the control it remains an open problem whether controllability can at all be decided by a finite number of differentiations of the data. This is closely related to the question whether the value function of the minimum-time optimal control problem is Hölder continuous. This paper briefly surveys some of the historical background behind the questions, and then presents recent progress, with focus on some unexpected properties of custom-designed polynomial systems.**

***Keywords*— Nonlinear control, controllability, value function, Hölder continuity**

## I. INTRODUCTION AND ACCESSIBILITY

Controllability is a fundamental property of control systems. One naturally asks for algorithmic criteria that allow one to decide controllability by a finite number of elementary computations. In the case of linear systems $\dot{x} = Ax + Bu$ defined by a pair of matrices $(A, B) \in (\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m})$, the Kalman rank condition guarantees that controllability is determined by the rank of the compound matrix $(B, AB, A^2B, \ldots A^{n-1}B)$.

In the case of nonlinear systems one distinguishes various different notions of controllability. Here we only consider finite dimensional systems that are affine in the control

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

and are defined by a finite collection of smooth vector fields $f, g_1, \ldots g_m$ on a smooth manifold, and locally integrable controls $u$ that take values in a (usually compact convex) subset $U \subseteq \mathbb{R}^m$. Later we will impose further regularity conditions, or conditions on the geometry on the set $U$ of admissible values of the control. Denote by $\mathcal{R}(T, p)$ the reachable set from $p$ at time $T$, that is, the set of all endpoints of trajectories starting at time $t = 0$ at $x(0) = p$ and corresponding to admissible controls $u\colon [0, T] \mapsto U$. The system (1) is called accessible from $p$ if for every $T > 0$ the reachable set $\mathcal{R}(T, p)$ has nonempty interior. It is well-known that in the case of analytic systems (e.g. when the set $U$ contains an open neighborhood of the origin), a necessary and sufficient condition for accessibility is provided by the Lie-algebra rank condition (LARC) $\dim L(f, g_1, \ldots g_m)(p) = n$, i.e. the Lie algebra generates by the system vector fields spans the tangent space $T_pM$ at the point $p$.

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To facilitate precise statements about lengths of brackets, introduce the free Lie algebra $L(X, Y_1, \ldots Y_n)$ over a set $\{X, Y_1, \ldots Y_m\}$ of $(m + 1)$ pairwise distinct indeterminates. This free Lie algebra has a natural graded structure: for any multi-index $\alpha = (\alpha_0, \alpha_1 \ldots \alpha_m)$ with $\alpha_k \in \mathbb{Z}_0^+$, write $L^\alpha(X, Y_1, \ldots Y_m)$ for the linear subspace of $L(X, Y_1, \ldots Y_n)$ spanned by all iterated Lie brackets that contain $\alpha_0$ copies of $X$ and $\alpha_k$ copies of $Y_k$. Also, for $N \in \mathbb{Z}_0^+$ write $L^N(X, Y_1, \ldots Y_m)$ for the direct sum of all homogeneous subspaces $L^\alpha(X, Y_1, \ldots Y_m)$ such that $|\alpha| = \alpha_0 + \alpha_1 + \ldots + \alpha_m = N$. Employ the suggestive notation $L^\alpha(f, g_1, \ldots g_m)$ and $L^N(f, g_1, \ldots g_m)$ for the respective images under the Lie algebra evaluation homomorphism that maps the indeterminates $X$ and $Y_k$ to the vector fields $f$ and $g_k$. Note that in general the subspaces $L^\alpha(f, g_1, \ldots g_m)$ have nontrivial intersection, and hence it does not make sense to talk about the length of an iterated Lie bracket of vector field $f$ and $g_i$.

With this terminology in place, we observe that if the system (1) is accessible, then there must exist a finite integer $N \in Z^+$ such that $\dim L^N(f, g_1, \ldots g_m)(p) = n$, i.e. Lie brackets of length at most $N$ already span $T_pM$.

However, unlike in the case of linear systems, it is impossible to determine the number $N$ a-priori, e.g. in terms of the dimension $n$ of the state-space. While the right hand side of (1) might involve arbitrarily high powers, one intuitively might expect that the *leading terms* of the components of the vector fields (when written in coordinates) give some bound on the $N$ which might work. But as the following simple example shows, this is not the case:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_1 e^{x_2} + x_1^s. \end{aligned} \tag{2}$$

Expanding the first term on the right hand side of the third equation $x_1 e^{x_2} = x_1 + x_1 x_2 + \frac{1}{2} x_1 x_2^2 + \frac{1}{6} x_1 x_2^3 + \ldots$ might suggest that some low order Lie brackets may establish accessibility. Yet one may easily verify that $\dim L^N(f, g)(0) < 3$ for all $N \leq 2s$.

This is easily explained by the observation that as geometric properties, accessibility and controllability are invariant under coordinate changes. In this case the triangular coordinate change $(y_1, y_2, y_3) = (x_1, x_2, x_3 - e^{x_2})$

transforms the system (2) into the form

$$\begin{array}{rcl}\dot{y}_1 & = & u \\ \dot{y}_2 & = & y_1 \\ \dot{y}_3 & = & y_1^s.\end{array} \tag{3}$$

which immediately explains why the first term $e^{x_2}x_1$ did not play any role for accessibility.

## II. NO A-PRIORI BOUNDS

There is little else to be said about accessibility at this stage. For analytic systems it is always determined by a finite number of differentiations, but one generally may not be able to predict how many differentiations are necessary for a positive answer.

However, unlike in the case of linear systems, for nonlinear systems accessibility does not imply controllability. Among various commonly used notions of nonlinear controllability, the following is among the best studied. Its importance is closely related to the fact that it in some sense is dual to optimality: The basic dichotomy is whether a reference trajectory lies on the boundary or in the interior of a the reachable sets. Hence necessary conditions for one translate into sufficient conditions for the other, and vice versa. More specifically, if $p$ is an equilibrium point of the system, e.g. if $f(p) = 0$ and $0 \in U$, then the system is small-time locally controllable (STLC) about $p$ if for every $T > 0$ the reachable set $\mathcal{R}(T, p)$ contains the point $p$ in its interior. More generally, if the system is small-time locally controllable (STLC) about a reference trajectory $x(\cdot, u^*)$ if for every $T > 0$ the point $x(T, u^*)$ is contained in the interior of the reachable set $\mathcal{R}(T, p)$ at time $T$.

The following simple example illustrates the distinction between deciding accessibility and controllability.

$$\begin{array}{rcl}\dot{x}_1 & = & u \\ \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_1^{2k} - x_2^{2s}.\end{array} \tag{4}$$

This system is accessible because $\dim L^{2k+1}(f, g)(0) = 3$, but it is controllable (STLC) if and only of $s < k$ – the coordinate-free Lie algebraic criterion thus requires consideration of vector fields in $L^{4s+1}(f, g)$, and generally, i.e. when $4s + 1 > 2k + 1$, requiring one to go beyond the order required to decide accessibility.

It is known that from a computational complexity point of view controllability is harder to decide than accessibility (it is NP-hard under the usual specifications such as rational data) [14], [18]. Nonetheless, much work since the 1980s by many researchers including Hermes, Sussmann, Stefani, Bianchini, Bressan, this author, and many others focused on finding algorithmic rank-like conditions similar to the LARC that would be necessary and sufficient for STLC, e.g. [5], [6], [8], [9], [12], [16], [17], [20], [21] and the references therein. The search for such conditions continues to this date, see e.g. [10] for most recent work.

The expectation that guided much classical work was that in some sense the leading terms in some Taylor-like expansion should determine controllability. The geometric character of the Lie algebraic approach easily detects terms such as in system (2) that are irrelevant for controllability. For driftless systems (i.e. when $f \equiv 0$), or more generally for systems that have an odd symmetry in the control (e.g. are invariant under the symmetry $(x, u) \mapsto (-x, -u)$, accessibility is easily shown to imply controllability. However, in general accessible systems one may expect possible *obstructions* that in the easiest case are due to even powers in a coordinate representation as in the example (4). As was first shown [20] in the example

$$\begin{array}{rcl}\dot{x}_1 & = & u \\ \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2^2 + x_1^3.\end{array} \tag{5}$$

the lowest order terms in a Taylor expansion, here $x_2^2$, need not be the leading terms for the purpose of controllability, here $x_1^3$. Instead, one considers multi-parameter scalings of the controls by

$$u_{\varepsilon,\delta}(\delta t) = \varepsilon u_{1,1}(t) \tag{6}$$

which in the multi-input case is to be read as the vector $(\varepsilon_1 u_1, \ldots \varepsilon_m u_m)$, etc.) These scalings correspond to families of control variations that traditionally were parameterized by the magnitude of the variations (like in classical calculus of variations), or by the length of the time intervals on which they difference from the reference control (*needle variations*), or a combination thereof. The multi-scaling of the controls gives rise to a notion of homogeneity for functions on the state space. More specifically, $\varphi\colon \mathbb{R}^n \mapsto \mathbb{R}$ is homogeneous of degree $(k, \ell)$ if along solution curves of the system $\varphi(x(\delta t, u_{\varepsilon,\delta})) = \delta^k \varepsilon^\ell \varphi(x(t, u_{1,1}))$ for all $t \in [0, T]$ and all $\varepsilon, \delta > 0$ sufficiently small. In turn, the type $(k, \ell)$ of a term $\varphi$ corresponds to the minimal number of factors $f$ and $g_i$ in any iterated Lie bracket $f_\pi \in L^{(k,\ell)}(f, g)$ of the system vector fields such that $(f_\pi \varphi)(0) \neq 0$. This scheme allows for choice of possible partial orderings, and it provides a framework for many conditions for STLC that were obtained in the 1980s.

Closely related are nilpotent approximating systems, compare e.g. [4], [9], which are constructed from the above ingredients, employing groups of dilations. Compare [15] for coordinate free, geometric description of groups and dilations and their applications to controllability and stabilization. More specifically, a system of form (1) is called nilpotent, if the Lie algebra $L(f, g_1, \ldots g_m)$ is nilpotent, i.e. there exists $N \in \mathbb{Z}^+$ such that for all $s > N$, $L^s(f, g_1, \ldots g_m) = \{0\}$. For such systems it is always possible to find local coordinates such that in these coordinates the vector fields $f$ and $g_k$ have a polynomial cascade form, i.e. $\frac{\partial}{\partial x_k}\langle\, dx_i\,,\, f\,\rangle \equiv 0$ and $\frac{\partial}{\partial x_k}\langle\, dx_i\,,\, g_j\,\rangle \equiv 0$ for all $k \geq i$ and all $j$ [13].

Explicit algorithms, e.g. [4], [6], [9], [19], yield nilpotent approximating systems (on the same state-space) that are designed to preserve the controllability properties of the original system. The key steps are finding adapted local coordinates, and then truncating Taylor expansions of the components of the vector fields $f$ and $g_k$ at appropriate orders. But as illustrated below by the examples (14) and (15), such algorithms generally will fail capture more delicate controllability features. Indeed, when looked at this from the other side one may argue that practically all general conditions for controllability (STLC) are based on using nilpotent approximating systems! Thus generally one expects that if the original system is STLC by virtue of the known sufficient conditions for STLC, especially Sussmann's general theorem [21], then the nilpotent approximating system is STLC. However, since the known necessary and the known sufficient conditions for STLC are not sharp, one may expect that generally any nilpotent approximations may either fail to preserve controllability, or even worse, be controllable even when the original system is not! We illustrate such systems in the subsequent sections.

## III. OBSTRUCTIONS TO CONTROLLABILITY

If one encounters brackets which give *one-sided new directions* these may be *possible obstructions* to controllability due to positive definite terms (all-even powers such as in (5). As illustrated in this system, such obstructions may possibly be neutralized by other terms that are sufficiently large (i.e. low order with respect to time and control size) to overcome the sign-definiteness. Necessary conditions for controllability such as in [6], [11], [19] provide upper bounds for the lengths of other brackets that possibly can neutralize possible obstructions – but such necessary conditions are only known for very few known possible obstructions. This leads one to hope that although an unpredictable number of differentiations (Lie brackets of unpredicted order) may be required to decide accessibility, given that a system is accessible it may only require a predictable number of additional differentiations to conclusively decide controllability.

*Conjecture 1:* There exists a function $N_2\colon \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ such that for every system of form (1) with $m = 1$ and $U = [-1, 1]$, if $\dim L^{N_1}(f, g)(0) = n$, then STLC can be decided by considering the values of the elements of $L^{N_2(N_1)}(f, g)$.

In words, if the iterated Lie brackets of length at most $N_1$ guarantee accessibility, then STLC can be decided by only investigating iterated Lie brackets of length at most $N_2(N_1)$. The existence of such a functions $N_2(N_1)$ would imply positive answers (for the special case of systems of form (1)) to an open problem posed by Agrachev [2]. in the first volume on Open Problems in Control Theory [1].

For illustration, consider the following system which in some sense is *critically* uncontrollable when $q(x) = 0$ for some critical value of $C > 0$, and thus might serve as a candidate counterexample.

$$\begin{aligned} \dot{x}_1 &= u \qquad |u(\cdot)| \le 1 \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_1^2 x_2^2 - C x_1^6 + q(x). \end{aligned} \tag{7}$$

To show that for $q(x) = 0$ and $C \le \frac{2}{15}$ the system (7) is not STLC about $x = 0$. Introduce the $C^1$ function $\phi\colon \mathbb{R}^3 \mapsto \mathbb{R}$

$$\phi(x) = x_3 + \tfrac{1}{3} x_1^3 x_2 \, |x_2| + \tfrac{2}{15} x_1^5 x_2. \tag{8}$$

A straightforward calculation establishes that the time derivative of $\phi$ along solutions of (7) satisfies

$$\tfrac{d}{dt}\phi(x) = (|x_2| + x_2 u)(|x_2| x_1^2 + \tfrac{2}{3} x_1^4) + (\tfrac{2}{15} - C) x_1^6 \tag{9}$$

Since $|u(\cdot)| \le 1$, it is clear that $\frac{d}{dt}\phi(x) \ge 0$ and hence the reachable set from $x_0 = 0$ is contained in the set $\{x \in \mathbb{R}^3 \colon \phi(x) \ge 0\}$, which does not contain an open neighborhood of the origin.

On the other hand, it is intriguing that part of the boundary of this reachable set of this system is *foliated* by periodic Pontryagin extremals. Indeed, choosing the feedback law $u(x) = \operatorname{sign}(x_2)$ renders $\phi$ constant along solution curves of the system (7) with $q(x) = 0$. In some sense, for $C = \frac{2}{15}$, while the term $C x_1^6$ is not able to *override* the term $x_1^2 x_2^2$, it manages to *neutralize* it, at least along this family of extremal curves. Thus one might conjecture that if one adds any, possibly much higher order, perturbation $q(x)$ whose average is negative along these periodic extremals, then one might be able to obtain controllability.

For simplicity, suppose $q(x) = -|x_1|^{2N}$ and again consider the feedback law $u(x) = \operatorname{sign}(x_2)$. The projections of the solution curves onto the $x_1 x_2$-plane are periodic and for some values of $M, t_0 \ge 0$ satisfy $x_2(t - t_0) = \pm\frac{1}{2}(M^2 - (x_1(t - t_0))^2)$ and $-M \le x_1(t - t_0) \le M$. Consequently, over any time interval of length $4M$, the change in the $x_3$-coordinate along the solution curve is readily calculated to be

$$x_3(t_1 + 4M) - x_3(t_1) = -4 \int_0^M t^{2N} dt = \tfrac{-4}{2N+1} M^{2N+1} < 0 \tag{10}$$

which is strictly negative. However, in order to reach such a periodic orbit from the origin, and also to get back to the $x_3$ axis, one has to employ controls that will let $\phi$ increase. The *optimal* choices use $u \equiv 1$ for initial and final time intervals of length $M/\sqrt{2}$. They transfer the state from $x_0 = 0$ to $x(M/\sqrt{2}) = (M/\sqrt{2}, M^2/4, x_{31})$ and from $(-M/\sqrt{2}, M^2/4, x_{32})$ to $(0, 0, x_{33})$. Calculate

$$x_{31} = x_{33} - x_{32} = \tfrac{23\sqrt{2}}{6720} \cdot M^7 + \tfrac{1}{2^N (2N+1)\sqrt{2}} \cdot M^{2N+1} \tag{11}$$

Concatenating the initial move to the periodic extremal, with $(n + \frac{\sqrt{2}}{4})$ loops, and a final transfer back to the $x_3$ axis requires a total time of $T = (4n + 2\sqrt{2})M$ and yields a terminal point at

$$x(T) = (0, 0, c_1 M^7 + (c_2 - n c_3) M^{2N+1}) \tag{12}$$

for suitable constants $c_i > 0$. Thus it appears to be a delicate question if it possible to make this quantity negative for arbitrarily small $T > 0$, by choosing $n \in \mathbb{Z}^+$ sufficiently large This necessarily makes $M > 0$ small. From the relation $T = (4n + 2\sqrt{2})M$ one finds that $M$ is inversely proportional to $n$, and hence for any integer $N$ such that $2N > 6$, sufficiently small $T > 0$ one will have $x_3(T) \geq 0$. This is even true in the critical case of $2N = 7$ where one needs to carefully consider the magnitudes of the various constants. In this case the result is clear when expressing the final $x_3$-value as an expression mixing $M$ and $T$:

$$x_3(T) = (\tfrac{23\sqrt{2}}{6720} - \tfrac{1}{8}T)\, M^7 + (\tfrac{17}{128} + \tfrac{\sqrt{2}}{8})\, M^8 \qquad (13)$$

which is positive for sufficiently small $T > 0$. Consequently for all $s > 3$ the system (7) with $q(x) = |x_1|^{2s}$ is not STLC, which, of course this is consistent with Stefani's necessary conditions for STLC [19].

What is intriguing about this example is even if an obstruction to controllability is fully neutralized in the sense of allowing families of periodic extremals on which these two brackets completely cancel each other, this is still not enough to alow for arbitrarily high perturbations to possibly provide controllability. Clearly, in this model system, it is the necessary transfer to and from the (due to the neutralization) free periodic orbits which prevents any higher order terms from providing controllability. It remains an open question if it possible to construct systems in which no such transfer is necessary – i.e. the free periodic orbits pass through the equilibrium point $x_0 = 0$.

## IV. FURTHER COMPLICATIONS

From an optimal control perspective, one may consider the value function $V\colon \mathbb{R}^n \mapsto [0,\infty]$ for the problem of reaching the origin in minimum time following trajectories of the time-reversed system. This function $V$ is continuous at the origin if and only if the system is STLC about $p = 0$. Moreover, $V$ is Hölder continuous of order $\frac{1}{N}$ at $p = 0$ if there exists a constant $C > 0$ such that for all sufficiently small times $t > 0$ the reachable set $\mathcal{R}(t,p)$ contains the open ball $B(0, Ct^N)$ about $p = 0$. Using the scaling (6) of the controls that is employed in classical control variations, one finds that the exponent $N$ is basically the same as the maximum of the lengths of the largest (formal) brackets required to ascertain accessibility and to neutralize all lower order possible obstructions to controllability. Yet these conditions are known to leave a wide gap between sufficient and necessary.

*Open problem 1:* (compare [2]) *If a system of form* (1) *with $U = [-1,1]^m$ is STLC does there exist an integer $N$ such that the value function $V$ of the minimum time problem is Hölder continuous of order $\frac{1}{N}$ at the origin.*?

The system (7) does not contradict an affirmative answer to this conjecture. However, the system

$$\begin{array}{rcl} \dot{x}_1 & = & u \\ \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_1^3 \\ \dot{x}_4 & = & x_3^2 + x_2^7 + q(x) \end{array} \qquad (14)$$

with gives renewed doubt to a positive answer. In [12] it was shown that this system with $q(x) \equiv 0$ is indeed STLC. Here the *higher order term* $x_2^7$ (which corresponds to brackets of type $\alpha = (7,8)$) was shown to be able to neutralize the lower order obstruction $x_2^3$ (which corresponds to brackets of type $\alpha = (6,3)$). A critical innovation in that work, needed to show that such unexpected domination was possible, was a new kind of families of control variations that basically is parameterized by the number of switchings of the primitive of the control. Indeed it can be shown that the system is not STLC if one restricts the set $\mathcal{U}$ of admissible controls to a cone (relative to the scaling (6)) over any compact set of controls. Furthermore it was shown in [12] that there exists a constant $C > 0$ such the reachable sets $\mathcal{R}(T,0)$ for all $T > 0$ sufficiently small contain the open balls about $x = 0$ of radius $CT^{57}$. While it has been elusive to show that the exponent 57 is the smallest possible, it is conjectured that already adding a nonzero definite term $q(x)$ of order higher than 15 can again destroy controllability.

This suggests that not only may the nilpotent approximating systems built according to the standard algorithms [9] fail to be STLC even when the original system is – as in the case of the system (14) with $q(x) \equiv 0$ – but it may also happen that similar approximations which include the terms such as the term $x_2^7$ which yielded STLC may be STLC even though the original system is not – e.g. in the original system such nonzero, higher order term $q(x)$ may override the neutralizing role of a term such as $x_2^7$.

This possibility of controllability status switching back and fourth as ever higher order terms are included in approximating systems arose in our study on needle variations that cannot be summed [7]. That lack of convexity of the set of infinitesimal needle variations is only possible in the setting of a nonstationary reference trajectory. In particular, consider the system

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & u_1 \\ \dot{z}_2 & = & u_2 \\ \dot{z}_3 & = & z_1^2 + (1 + u_{01}) \\ \dot{z}_4 & = & z_2^2 + (1 + u_{02}) \\ \dot{z}_5 & = & z_4 z_1^2 - C_1 z_1^7 + C_2(z_1^{10} + z_2^{10}) \\ \dot{z}_6 & = & z_3 z_2^2 - C_1 z_2^7 + C_2(z_1^{10} + z_2^{10}) \end{array} \right. \qquad (15)$$

with $(u_{01}, u_{02}, u_1, u_2)(\cdot) \in U = [-1,1]^4$ starting from $z = 0$ about to the reference trajectory $z^*(t) = (0,0,t,t,0,0)$ corresponding to $u^* \equiv 0$. It is easy to see that the system (15) with $C_1 = C_2 = 0$ is not controllable about $(z^*, u^*)$. In [7] it was shown that the system (15) with $C_1 = 1$ and $C_2 = 0$ is STLC about $(z^*, u^*)$, yet when adding another *higher order* perturbation by setting $C_1 = C_2 = 1$ then controllability is again lost. In this work we further investigate the growth behavior of the reachable sets of this system, and the associated regularity of the value function of the minimum-time problem. It remains an open question whether it is possible to construct infinite such chains of alternations of controllability and lack of controllability by successively including ever higher order terms.

## V. SUMMARY AND OUTLOOK

In summary, these studies provide some new insights, and in particular valuable intuition but they remain a long way from a conclusive answer to Agrachev's question.

*Open problem 2:* (compare [2]) *If a system $\Sigma$ of form* (1) *with $U = [-1, 1]^m$ is STLC does there exist an integer $N$ such that every system $\Sigma'$ of the same form is also STLC if the Taylor expansions of the vector fields of the two systems agree up to order $N$?*

The investigations and constructions discussed above illustrate possible mechanisms by which the reachable set may have a much slower growth rate (or the value function much less regularity) than the system data might make one expect. On the other hand, there does not seem to be a direct way to generalize these constructions to go from a finite number of alternations to an infinite number of alternations of the controllability status. It appears ever more likely that the question may only be resolved by using more abstract analytic reasoning, similar to the one in [3] which established final answers to the subanalyticity of the value function.

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