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Functions: Looking ahead beyond calculus

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The concept of a *function* is a unifying foundation of modern mathematics. Increasingly, the function concept also guides middle school algebra curricula, and ubiquitous research studies analyze its learning. Such recent changes have helped bridging the traditional gap between high-school algebra and college-level calculus. But teaching and learning functions in a narrow context opens and widens the rift between calculus and both post-calculus courses and real-world applications. This note provides several examples to illustrate that some of these efforts miss what mathematicians consider key characteristics of functions. It is based on the observation that these school curricular changes and the majority of research studies focus almost exclusively on (real valued) functions of a (scalar) continuous variable. We focus on composition as the distinguishing operation on functions, issues relating to domain and codomain, injectivity and surjectivity, and the interplay with alternations of universal and existential quantifiers. This note is to stimulate discussion about how much teachers of algebra and curriculum designers at all levels need to understand about functions themselves, and about how deeply should students at which level understand functions.

1 Introduction

This note is motivated by personal experiences when teaching a wide variety of post-calculus courses at a large public North American research university, by discussions with middle and high school mathematics teachers, and impressions at numerous colloquia and conferences on research in mathematics education. In a nutshell, the author welcomes the increased attention given in early grades to the fundamental and unifying concept of a *function*, partially implementing recommendations made in the *Principles and Standards of School Mathematics* [8]. It is a great improvement if middle school algebra teachers do not just teach algebra for the sake of teaching algebra, but instead if (s)he looks far ahead and understands the importance and meaning of the subject for all students, whether they study university level science or not. But the author is worried that the reformed curricula do not adequately develop

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some the most important characteristics of functions. This becomes particularly clear in discussions with teachers, curriculum designers and in numerous colloquia and conference talks where the focus is solely on (piecewise smooth, real-valued) functions of a (scalar) continuous variable. Worse yet, at numerous talks, the speakers, when questioned, did not even seem to understand why mathematicians would not be perfectly happy with what is taught under the name function at the earlier levels. As an instructor of courses named vector calculus (VC), differential equations (DE), linear algebra (LA), introduction to writing proofs, advanced calculus / introduction to analysis, partial differential equations, abstract algebra, and differential geometry, this author, semester after semester, encounters large groups of students who demonstrate almost fatal misunderstandings of what characterizes a function. All too often, this author wonders whether it might not have been better if these students' teachers had not used the word function at all, not caused misconceptions that are extremely hard to undo. These experiences are reminiscent of the foundational work [6, 7] that identifies the need to first wipe the slate clean by explicitly addressing common misconceptions in mechanics, before even starting with new material.

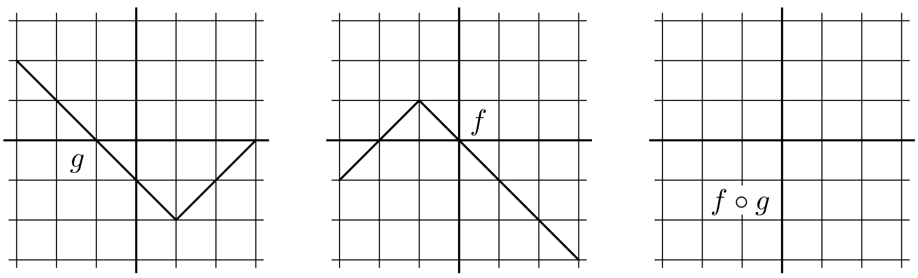
Paralleling the historical migration of other mathematics subjects from being university courses to becoming schools courses, calculus is becoming a high school course: In 2004 in the United States, 175,000 and 50,000 students took the advanced placement (AP) exams in calculus at the AB and BC levels, respectively [3]. Much larger numbers of students enroll in high school advanced placement courses without taking the exams. The success of the AP program is being analyzed in many studies, e.g. [10, 11]. This migration facilitates continuity between middle school algebra and calculus. But we fear that it may also be a cause for a widening chasm between calculus and follow-on subjects. This is exacerbated by calculus even at the universities now often being taught exclusively by full-time instructors, where in the past research faculty always had an eye on how to keep calculus connected with the next levels.

There are so many studies and publications on the function concept that it is impossible to list even a representative fraction in this place. We limit ourselves to a few remarks about closely related literature. Since the well-known article [18] many studies have analyzed various aspects of the function concept, applicable to the entire range from the elementary levels to beyond *introduction to analysis* (advanced calculus), see e.g. [1, 16, 17] and the references therein for particularly deep analysis. A large number of studies by others focus on real-valued functions of a single continuous variable with primary focus on the notion of covariation, see e.g. [2, 12]. While these are close to the immediate needs of teachers, other mathematicians are concerned that the resulting policy recommendations miss important features.

The subsequent sections provide examples and test cases from different contexts that illustrate shortcomings of students' understandings in selected features of functions: composition as the characteristic operation on functions, domain and codomain, preservation of structure (e.g. monotonicity and linearity), injectivity and surjectivity, invertibility, and alternating quantifiers. For lack of space we do not address the important interplay of function as taught in most mathematics courses with algorithms and functions found under various names such as *subroutine, procedure, method* in FORTRAN, ISETL, C, MATLAB, MAPLE, JAVA. As argued in other places, e.g. [4, 9, 17] the use of computer languages can be a powerful tool to construct stronger conceptual images of the function concept.

2 Composition

The set of functions Y^X from a set X to a set Y inherits many algebraic structures from the codomain Y. Even at advanced levels such as abstract algebra, many students have never consciously reflected on this basic observation. If the codomain Y is a ring or a vector space then the set of functions inherits this structure under pointwise operations. The true meanings of pointwise definitions of operations on functions such as $(f+g)(x) = f(x) + g(x)$ are often poorly understood. What distinguishes functions from the algebraic objects that students have encountered previously is that functions may be composed. Some aspects of compositions are readily taught, such as evaluating compositions (of formulas) at numbers, and basic symbolic manipulations. Students also quickly learn how to use the chain-rule for finding derivatives of expressions such as $\sin(x^2)$. But these are very limited levels of understanding of compositions. Some of the author's favorite questions on his *first-day-of-class diagnostic tests* are given below. The first question illustrates that even evaluation of compositions of functions is rarely mastered when combined with the need for even elementary graphical reasoning.



Question 2.1. Given the graphs of two functions f and g above sketch the graph of their composition $f \circ g$. Use a minimal number of function evaluations.

Generally poorly understood are more general properties of compositions of functions, as illustrated typical answers to:

Question 2.2. Suppose that f, g are monotonically decreasing functions. Decide whether the inverse f^{-1} and the composition $f \circ g$ are decreasing, increasing, or whether both are possible.

This question is applicable in the broader context of order preserving/reversing functions between partially ordered sets as commonly studied in post-calculus courses. The next task illustrates that chain (and product-)rule are often only mastered for specific explicit formulas, not for functions, hinting at a superficial understanding of compositions.

Question 2.3 Suppose that $f, g\colon \mathbf{R} \mapsto \mathbf{R}$ are differentiable. Use rules of differentiation to simplify $(f \circ g)''$.

Let us change gears, and consider the following three tables. The first represents the percentage of the recommended daily allowance (RDA) of vitamins A and C and iron for one serving each of various fruit. The second table lists the contents of two different kinds of dessert baskets offered at a cafeteria, the third lists the number of baskets that three students consumed today for breakfast, lunch and dinner.

	Orange	Apple	Kiwi	Banana
A	0	2	2	0
C	130	6	240	15
Iron	2	2	4	2

	Basket 1	Basket 2
Orange	1	1
Apple	1	0
Kiwi	0	1
Banana	0	1

	Ann	Bob	Jill
Basket 1	1	0	2
Basket 2	2	3	0

Where are the functions? What class are we in? There are many places where we consider tables and matrices as functions of two variables: The notation makes all the difference, we write $f(x, y)$ and a_{ij} but should be able to understand just as well $a(i, j)$ and f_{xy}. Alternatively, each row and each column in each table represents a function, as does each table in its entirety – think about the practical interpretation, the domain and the codomain!

A typical use of these tables in the first course of linear algebra is to answer: Why do we multiply matrices the way we multiply them? Most students at this level have learnt *how*. But who understands *why*? Given these tables *with* their explicit labels, it is an elementary school task to figure out what to do with them, how to meaningfully combine them into new tables that answer the obvious questions: How many percent of her RDA of vitamin A did Ann get from the fruit she ate today?

We have three tables – and hence we may multiply $(A \cdot B) \cdot C$ or $A \cdot (B \cdot C)$. Do we get the same result? Why? It was clear to my children that the resulting numbers must be the same, simply because each number has meaning. Yet from a linear algebra perspective, we want a more formal argument why matrix multiplication is associative. The mathematician thinks: multiplication by a matrix is a function, multiplication of matrices corresponds to function composition and function composition is associative – the most fundamental property of function composition. Are our students prepared to understand these functions? When their teachers teach functions, do they understand the fundamental importance of associativity of function composition? These tables also nicely illustrate the importance of domain and codomain – here we need matching dimensions (sizes), but even then some products are utterly meaningless (though algebraically legal). As an immediate corollary we get that matrix multiplication is not commutative.

It is challenging to teach students, who have never really thought of multiplication by a number as a function, that *multiplication by a matrix* is a function. All too often the notation $y = 3x$ gets in the way – where we really need to think *"times 3"*. Too many students think it cannot be a function if there is no x. Indeed, all too often the use of the letter x is an obstacle. Why not say that the derivative of sin is cos (MAPLE uses this language), or that the derivative of *"squaring"* is *"multiplication by 2"*? In the same vein, when in multi-variable calculus we create a table of values for the function $f(x, y) = xy$ it seems that our students never thought of multiplication (of two numbers) as a function – whose inputs are pairs of numbers. This is one of the first functions ever encountered. Do the teachers understand how to sow the seeds for a comprehensive understanding of the concept of functions?

3 Domain and properties of the image

Many students confuse a formula with a letter x with the concept of a function.

Question 3.1. Does $y = \begin{cases} 1 - x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ define a (*one*) function?

The majority of second year college students votes that this is two functions, not one. This becomes very painful in DE courses where natural forcing functions typically are piecewise smooth.

Question 3.2. Find the derivative of $y = \log(\log(\sin x)))$ with respect to x and overlay the graphs of y and y'.

Every single time that I asked my post-calculus students, the large majority of them correctly (?) applied derivation rules. Yet they did not even see the problem: What does it mean to differentiate a function with an empty domain? Even more curious, how can its derivative have a nonempty domain? But then the antiderivatives of the derivative also must have a nonempty domain – but how do these relate to the original? It is clear that these students confuse the (algebraic) derivate of a formula (symbolic algebraic expression) with the derivative of a function. Most introductory textbooks begin with well-intentioned pictorial and tabular examples of functions with discrete domain (and codomains) – yet what is learnt in these lessons is quickly superseded by an confusing functions with symbolic formulas. See [1] for a related discussion.

Students venturing beyond calculus have trouble recognizing functions when they appear as vector fields in vector calculus, and as differential operators in the first DE course. In the first case the problem is that not only is the input not a number, but a point in the plane, in 3-space, on a curve or a surface, but also its output is not a number, but a vector. In the second case, both the input and the output of the function (operator) are functions themselves.

In its most simple example, the step from the DE $y''+4y'+3y = g(t)$ to the differential operator $L\colon y \mapsto y'' + 4y' + 3y$ is as important and powerful as the step from the quadratic equation $x^2 + 4x + 3 = 0$ to the quadratic function $f\colon x \mapsto x^2 + 4x + 3$. The graph of the first consists of the two points -1 and -3 on the real number line, whereas the graph of the latter is a parabola. An easy way to recognize the power of this step is to consider the quadratic equation $x^2 + 4x + 2.9 = 0$. The first view offers no hint. From the second picture it is immediate that the solutions are approximately -3.05 and -0.95. (The x^2 term has unit coefficient, and hence the slope one unit to either side of the vertex is $\pm 2 \ldots$.) Precisely this kind of thinking is needed in the context of DEs. It is a huge step from the algebraic equation to the function. Do the teachers understand how important it is, and that it is repeated later, again and again?

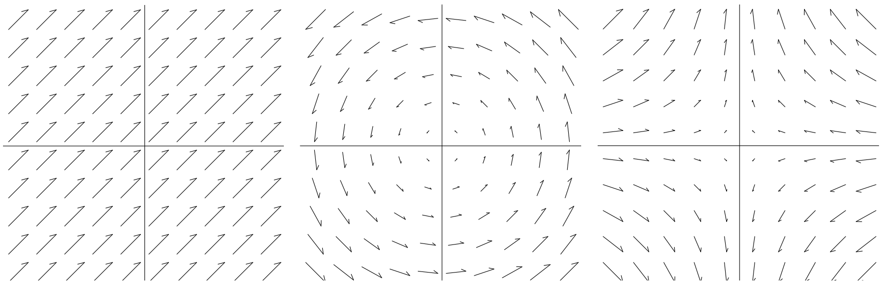
Many mathematicians prefer to only work with the vector field defined by a DE since it is a function (not an equation) and hence has much more desirable properties. At its simplest, there is a precise notion of equality: two functions f and g are equal if they have the same domains, same codomains, and if for every input their outputs agree. On the other hand, are the equations $y = x$, $x = y$, and $y - x = 0$ the *"same"*? On the algebra and calculus level teachers and students manage to muddle their way through: the right answer is the one in the back of the book. This misses a golden opportunity to *"sell"* functions (for their precision) to students clinging to equations, and to explicitly address the common quandary of *same function, different formula*.
For now the first DE course stays an *equations* course, with the function notions (painfully) employed only when absolutely needed as in the form of differential operators. Our students simply arrive with such a warped sense of function at this course, that it would take too long to wipe clean the slate before being able to start. The connection with vector fields – which our students do not recognize as functions – remains on the fringes, with consequent inability to

connect DEs to curl and divergence.

At all levels we would like our students to have a sense for the depth of the question which structures are preserved by which functions. Focusing too much on only (real valued) functions on the real line muddies the waters in a different way, as the real numbers simply have too much structure: Arithmetic properties, order properties, and topological properties (closeness). Often it is easier to study properties in more abstract settings as this allows one to focus on the essentials, what makes things work. This is beautifully accomplished in the new *Introduction to Analysis* textbook *Closer and Closer* [13].

Arguably the most important algebraic notions are linearity and multiplicativity. Of course, these may also be mixed, as, e.g., the exponential function maps the additive group $(\mathbf{R}, +)$ to the multiplicative group $(\mathbf{R}^+, \ \cdot\ )$, and hence its inverse allows one to use a *slide-rule* to perform multiplication by simple adding. For an in-depth study of teaching multiplicative properties at the school level see [15]. At the precalculus level students study linear functions and their graphs ad infinitum, and they also learn rules how not to simplify e.g. $\frac{77+6x}{7+3x^2}$, $\sqrt{x^2+9}$, $\sin(\alpha+\beta)$. But do they and their teachers understand the fundamental importance of *linearity*, understand the importance of differentiation and integrations being linear?

Question 3.3. Which of the vector fields depicted below is linear?



Even among university faculty the answers are all too often false (correct: the first is not linear, the other two are linear). The universal definition of linearity is: A function $f\colon X \mapsto Y$ is linear if for all scalars c and all $p, q \in X$, $f(cp+q) = cf(p) + f(q)$. (For this to make sense, one must be able to add and take scalar multiples in both domain and codomain, i.e. both need to be vector spaces, or modules.) In their first year after calculus, students encounter this in linear algebra, DEs, mechanics (superposition principle), linear electric circuits, ... Do the teachers of linear functions understand that $y = mx + b$ is a lot more than just one step before studying more complicated formulas? A recent parent-teacher meeting with my daughter's algebra teacher again indicates otherwise. Although he proudly told her that in 7th grade she would *"learn about functions, a concept that will get her prepared for calculus"*, he had no understanding of the role of functions outside calculus.

Next are *increasing*, *decreasing*, *convex* and *concave* functions – well understood to be a core calculus topics. All too often students *define* [?] a function to be increasing (or convex) if its (second) derivative is positive. This is the first hint that increasing (convexity) have not been taught properly. Another hunch is the earlier question 2.2 whether inverses and compositions of decreasing functions are decreasing. Monotonicity comes in many different forms, at many levels – e.g. the modern mathematical models of many biological systems are rooted in the theory of *monotone systems of DEs* [14]. Likely the best preparation for later uses is a strong verbal understanding of *increasing* and *increasing at a decreasing rate*. We would like to see more familiarity of *order preserving/reversing* functions on partially ordered sets. Well-tested activities start with directed graphs that represent tournament results / rankings.

Another key property is continuity, called *stability* in the applied sciences, loosely defined as: small changes in the input yield small changes in the output.

Question 3.4. Using this figurative notion, decide whether $f\colon \mathbf{R}\setminus\{0\} \mapsto \mathbf{R}$, $f(x) = \frac{1}{x}$ and $g\colon \cup_{n=-\infty}^{\infty}(2n, 2n+1) \mapsto \mathbf{R}$, $g(x) = \log(\sin x)$ are continuous.

Every mathematician will agree that both functions are continuous – the figurative criterion matches the precise topological / analytical definitions. Yet, the commonly used criterion of being able to draw the graph of the function without lifting the pencil from the paper fails – due to lack of *connectedness of the domain.* This is a delicate topic. What should the secondary school teacher know, and what should (s)he tell her/his students?

4 One-to-one, quantifiers and related topics

The question whether one can go back, *undo* a function, leads one to analyze when functions are one-to-one (injective) and onto (surjective). Most textbooks include a few pictures of functions between discrete sets to illustrate these concepts – but due to lack of follow-up and making repeated connections to other settings, these lessons seem to have little lasting impact.

It has been sown that it is possible to successfully teach secondary school students functions and injectivity in the precise language of alternating universal and existential quantifiers (and negations of quantified statements). But in most school classes one finds years of very casual treatment of the precise definitions of a function and of injectivity. We advocate that all secondary school mathematics teachers have a profound understanding of functions and of injectivity so that they can conduct regular activities that promote thinking along the lines of modern mathematics, thereby opening the pipeline to more students, and cushioning the shock when students step outside calculus.

In repeated courses of teaching *Introduction to Mathematical Reasoning*, the author has found it beneficial to sidestep real-valued functions of a real variable for this purpose, as the majority of students just carries too much baggage of misconceptions and misunderstandings about functions. At some time these will have to all be addressed, and painstakingly cleared up – this is not unlike the situation in university physics described in [6,7]. Together with colleagues who taught the same material at the school level to both teachers and students, we found counting functions to be a particularly amenable area for developing notions of function, one-to-oneness and bijection. These are bijective functions whose domain is either the set of positive integers or an interval $\mathbf{Z} \cap [1, n]$. We explicitly count permutations, combinations, pairs of numbers, rationals, and arrangements of playing cards (both sets and sequences) – which nicely show the mathematics behind stunning *card tricks*. Algebraic formulas are obtainable – but often these are unattractive. The focus is on the structural properties that the constructed objects are indeed (counting) functions (one-to-one and onto). Our experiences indicate that many of these activities are just as well suited for younger students at the schools level – and they would allow students at an early age to develop profound and broad understanding of the central concepts that characterize functions, as opposed to the currently prevalent tendencies that solely emphasize covariation of scalar continuous variable with a narrow focus on conceptual understanding of single-variable calculus.

From a more advanced perspective, functions are routinely employed, especially in analysis, to simplify statements that otherwise are hard to read due to large numbers of alternations of existential and universal quantifiers. A typical definition from calculus reads: *"A function $f\colon \mathbf{R} \mapsto \mathbf{R}$ has a limit at $a \in \mathbf{R}$ if there exists a number $L \in \mathbf{R}$ such that for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $x \in \mathbf{R}$, if the distance from a to x is less than δ then the distance of $f(x)$ from L is less than ε".* Using functions this can, for general metric spaces X and Y, be rewritten as: *"A function $f\colon X \mapsto Y$ has the limit $L \in Y$ at $a \in X$ if there exists a continuous, strictly increasing function $\varepsilon\colon \mathbf{R}_0^+ \mapsto \mathbf{R}_0^+$, with $\varepsilon(0) = 0$ such that for all $x \neq a$, $d(f(x), L) < \varepsilon(d(x, a))$."*

A particularly successful notion is that of a $\mathcal{KL}$-function which has a similarly simple definition and which allows one to avoid complicated alternations of quantifiers in the characterization of *asymptotic stability* of a dynamical system. For lack of space we cannot elaborate the details here, but encourage the reader to contact the author who in turn plans to give a detailed description of the use of functions to encapsulate alternating quantifiers, thereby making otherwise too complicated statements understandable.

5 Summary and conclusion

In many places efforts are made to teach and develop the concept of function at earlier levels, while research studies analyze the learning of this concept. This note gives examples that suggest that some of these efforts focus on a very narrow slice of this concept, leading to far from desirable levels of understanding of functions at various levels.

Sometimes it makes sense to start with one special case, and gradually generalize a concept. In other cases harm is done by too much focussing first on a special case – the exclusive study of linear DEs being a case in point where students confuse what remains true and which methods don't apply to nonlinear problems. We leave it for discussion in which of these two ways to introduce functions, and which aspects of functions should be studied at which level. A follow-up task is to agree on model activities that develop currently neglected critical aspects of functions at various levels. This requires a close cooperation between curriculum developers at the schools level, education researchers, and faculty with broad teaching experiences at postcalculus levels.

References

[1] Akkoç,H., and Tall, D. (2005 preprint). A Mismatch between curriculum design and student learning: the case of the function concept. To be presented at Brit. Colloq. Math. Educ., Warwick.

[2] Carlson, M. (1998), A cross-sectional investigation of the development of the function concept. Res. in Coll. Math. Educ. III, CBMS, Issues Math. Educ. **7**; Amer. Math. Soc, 114–162.

[3] College Board, (2005), Advanced placement report to the nation. `http://www.collegeboard.com/prod_downloads/about/news_info/ap/2005/ap-report-nation.pdf`

[4] Dubinsky, E., and Harel, G. (1992), The concept of function, aspects of epistemology and pedagogy. *MAA Notes* **25**, 85–106.

[5] Hahn, W., (1967), *Stability of Motion.* Springer, Heidelberg.

[6] Hestenes, D., Wells, M., and Swackhamer, G. (1992), Force concept inventory. *The Physics Teacher*, **30** 141–151.

[7] Hestenes, D., and Wells, M. (1992). A mechanics baseline test. *The Physics Teacher*, **30** 159–166.

[8] Nat. Council of Teachers of Math., (2000). Principles and standards of school mathematics.

[9] Kawski, M., (2003), Functions and operators in MAPLE and in MATLAB. *Proc. 4th Southern Hemisphere Symp. on Undergrad. Math. Teaching.*

[10] Klopfenstein, K., (2004) Recommendations for maintaining the quality of advanced placement programs. *American Secondary Education* **32**, 39–48.

[11] Klopfenstein, K., and Thomas, K., (2005, preprint) The link between advanced placement experience and college success. `http://www.utdallas.edu/research/tsp/Index.htm`

[12] Saldanha, L., Thompson, P. (1998), Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. *Proc. Ann. Mtg. PME North America.*

[13] Schumacher, C., (2005, to appear), *Closer and closer.* `http://www.kenyon.edu/x6033.xml`

[14] Smith, H., (1995), *Monotone Dynamical Systems, an Introduction to the Theory of Competitive and Cooperative Systems.* Amer. Math. Soc., Math. Surveys Monographs.

[15] Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., and Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, **1**, 127-155.

[16] Tall, D., M., DeMarois, P., (1999) Function: organizing principle or cognitive root? In O. Zaslavsky (ed.), *Proc. 23rd Conf. PME,* **2**, 257-264.

[17] Tall, D., McGowen, M., DeMarois, P., (2000) The function machine as a cognitive root for building a rich concept image of the function concept. *Proc. of PME-NA,* **1**, 247-254.

[18] Thompson, P. (1994). Students, functions, and the undergraduate curriculum. in Dubinsky, E., Schoenfeld, A., and Kaput, J., *Res. Coll. Math. Educ. I*, CBMS Issues in Math. Educ., **4**, 21-44.