Calculus of Vector Fields using JAVA

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Abstract. We present model software that invites a unified, highly interactive, visual approach to differential and integral vector calculus, and differential equations. The tool is free and requires only a JAVA-enabled WWW-browser.

This project started with the simple question: *"If zooming is so much better for understanding derivatives in first year calculus* (than the traditional secant lines which become tangent lines), *then why not zoom on vector fields to study the curl and divergence?"* This quickly led to the next objectives: Connect the divergence and curl of vector calculus to differential equations. Provide tools for line and flux integrals that connect differential and integral calculus, and lay the groundwork for the integral theorems.

The implementation into JAVA exhibits numerous features that are desirable for mathematical software in many areas. Foremost they are its *visual language*, its interactivity, and versatility which invites further exploration and discovery that go far beyond repeating *canned experiments*.

1 Introduction

We demonstrate how modern technology for interactive visualization can completely transform the learning, teaching, and understanding of college level core mathematics.

Historically, vector calculus is known for its abundance of forbiddingly complicated algebraic formulas. Few learners gain an in-depth understanding of the core concepts. Yet these are becoming ever more important to an ever broader group of scientists, far beyond the traditional users in electro-magnetics and fluid mechanics. Just think of the modern cardiologist who needs to have at least an intuitive understanding of turbulence and similar characteristics of blood flow in coronary arteries.

The *Vector Field Analyzer II*, short *VFA II*, invites a completely new approach to the core topics of both the differential and integral calculus of vector fields. The tool is freely available on the WWW, requiring no more than a JAVA-enabled browser. `http://math.la.asu.edu/~kawski/vfa2/comments.html` An accompanying work-book and textbook are under preparation. In addition to providing some immediate practical utilities for learning, teaching and understanding vector calculus, the VFA II also serves as a model for a new class of interactive software. Some of the key features are

- The visual language of the VFA II pushes the traditional algebraic symbols deep into the background.
- The VFA II is destined for true experimentation – inviting explorations far beyond any *canned* experiments.
- While organized along three separate panels, the VFA II promotes deep conceptual linkages between different parts of a traditionally very fragmented mathematics curriculum.

The plan of this presentation is to let most of the audience experience the role of the learner, discovering many new views of vector calculus.

The VFA II has many hidden features that go far beyond what is possible to explore in a short presentation. These include topologies other than that of the plane, different representations for covariant and contravariant vector fields, curves defined symbolically, local versus global scaling, aliasing effects versus continuity, etc.

Aside from featuring a sophisticated parser that allows the user to enter very diverse set of algebraic expressions, the VFA II works purely numerically. It does not do any symbolic computations. The numerical algorithms employed are *general purpose* selected for being sufficiently robust to allow for experiments far beyond the original purpose, but they lack the sophistication of e.g. *symplectic integrators* that could e.g. guarantee computed periodicity. The objective is to provide a useful tool at the college level that is based on sound mathematics – but without becoming unnecessarily worried about sophisticated advanced notions. E.g. the approach taken here is very much based on the idea that *differentiability* means *approximability by linear objects* – in turn it does not worry about modern differential geometry might object to an intrinsic notion of a *linear field*, or how to compare *tangent vectors* based at different points (i.e. the VFA II works with the *trivial connection.*

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2 Example: Derivatives of vector fields via zooming

Suppose $\vec{F}(x, y)$ is a vector field in the plane. We would like to see its derivative by appropriately zooming in at various points. The naive zooming in on the vector field at a point (x_0, y_0) – use the radio-button **Contin** – only magnifies the domain. The resulting image in the lens shows a constant vector field which is *the right picture* for studying continuity and integrability (Euler's and Runge-Kutta-like methods for the associated differential equation and for line integrals). However, to see the derivative we need to analyze the difference $\vec{F}(x, y) - \vec{F}(x_0, y_0)$ for (x, y) near (x_0, y_0). The rescaling is automatic. Choose the button **Deriv**. After sufficient magnification of both domain and range, the resulting image in the lens shows the linear field (using $(\Delta x, \Delta y)$ as coordinates inside the lens):

$$DF_{(x_0,y_0)}(\Delta x, \Delta y) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(x_0, y_0) & \frac{\partial F_1}{\partial y}(x_0, y_0) \\ \frac{\partial F_2}{\partial x}(x_0, y_0) & \frac{\partial F_2}{\partial y}(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

It is unfortunate that many traditional vector calculus classes *forget* to properly study linear fields before proceeding to derivatives. We have found that it is easy and most beneficial to invest in a detailed study of linear fields – analogous to studying linear functions and lines before attempting calculus. Indeed linear vector fields are extremely well-suited to learn *homogeneity* $L(cp) = cL(p)$ and *additivity* $L(p+q) = L(p) + L(q)$. Look inside the lens and check for linearity!

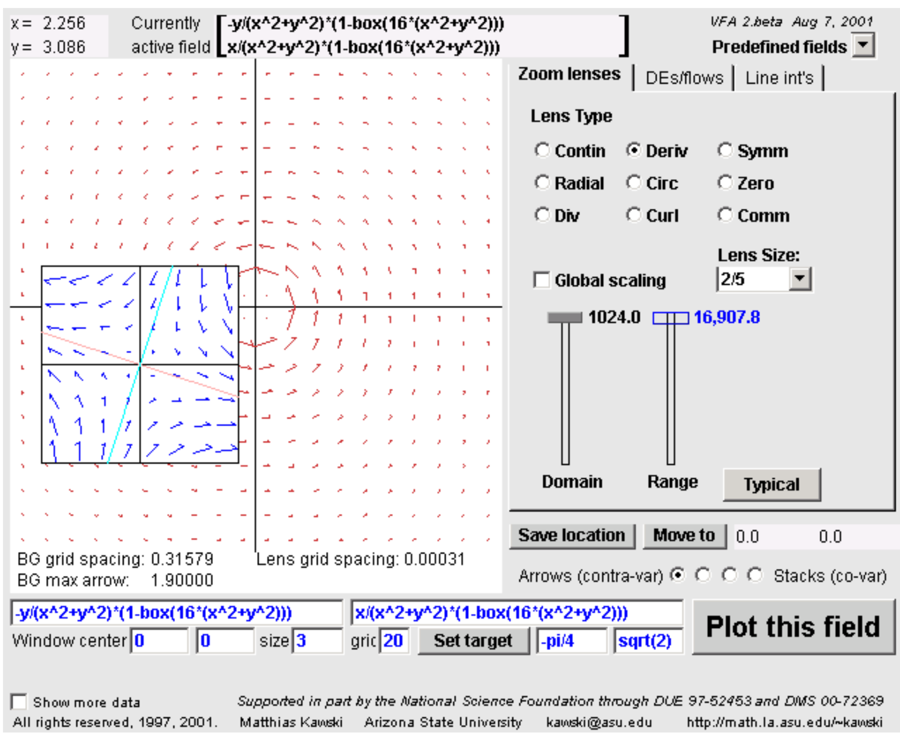

Figure 1: The derivative of the irrotational magnetic field

Differentiability is defined via approximability by a linear vector field – the derivative at that point. A quick check for understanding: What is the derivative of a linear field? Try **Predefined fields**, **Harmonic oscillator** and zoom for its derivative at various points. If confused – recall what you see when zooming in on a straight line in the fist calculus course. The common difficulty in multi-variable calculus is that the derivative has two arguments: It is a linear function of the increment (dx, dy), though it generally depends nonlinearly on the point (x_0, y_0).

Many applied sciences do not need to whole derivative DF but they care primarily about its *geometric* components: the divergence $\text{div}F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$ (which is the *trace* of DF), and the rotation (or scalar curl)

$\mathrm{rot}F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ (the *skew symmetric* part of DF. The VFA II provides an array of lenses that are easily selected, e.g. **Div** and **Curl** and which show the corresponding local rates of expansion or contraction, or the local rates of rotation / spinning.

More advanced investigations analyze the derivatives of the vector field $(\mathrm{Re}f(x+iy), -\mathrm{Im}f(x+iy))$ associated with a complex analytic function $f(z)$. The Cauchy-Riemann equations imply that such vector fields are both irrotational and divergence free. Hence the derivative lens will show a linear field that corresponds to a symmetric (and hence orthogonally diagonalizable) matrix of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ where the angle α depends on the point (x_0, y_0) and determines the orientation of the eigenspaces of the derivative.

3 Example: Connecting vector calculus and differential equations

Traditional classes and textbooks in vector calculus and differential equations often use very different algebraic symbols. Hence it is of little surprise that most students (and many teachers, too) do *not make the connection*. The interactive visual language of the VFA II is opposite: One can't even tell into which class the picture belongs!

Many software packages calculate and animate the solution curves of systems of differential equations. But this is not enough to make the connection with divergence and curl. The key is to consider entire *regions* of initial conditions. The integral of the divergence determines the growth of the area/volume of this region. The integral of the curl determines the rotation.

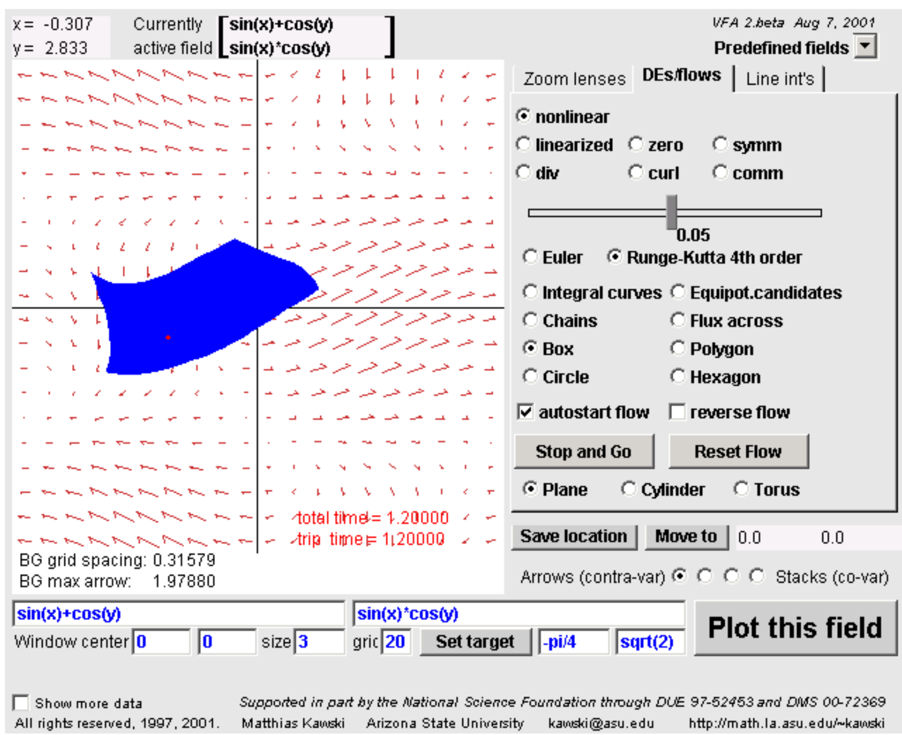

Figure 2: A region of initial conditions acted upon by a nonlinear flow

The VFA II provides a diverse set of tools to study various aspects of such flows. Interpreting the field as a velocity field one may investigate either flow lines (integral curves, trajectories) of even very large sets of points or even the evolution of entire regions of initial conditions. Corresponding to the collection of zoom lenses, the VFA II provides matching choices for different aspects of the flow: In addition to considering the full **Nonlinear** flow of the field, one may also view only the **Linearized** flow (about the trajectory followed by the *center of mass* of the original region). Other options include viewing the integral of the divergence, **Div**, which shows only the area/volume change, and the integral of the skew-symmetric part of the linearization

Curl, which as an orthogonal map preserves both area and angles. This latter may be interpreted as showing the rotation of an infinitesimally small rigid body subject to the flow.

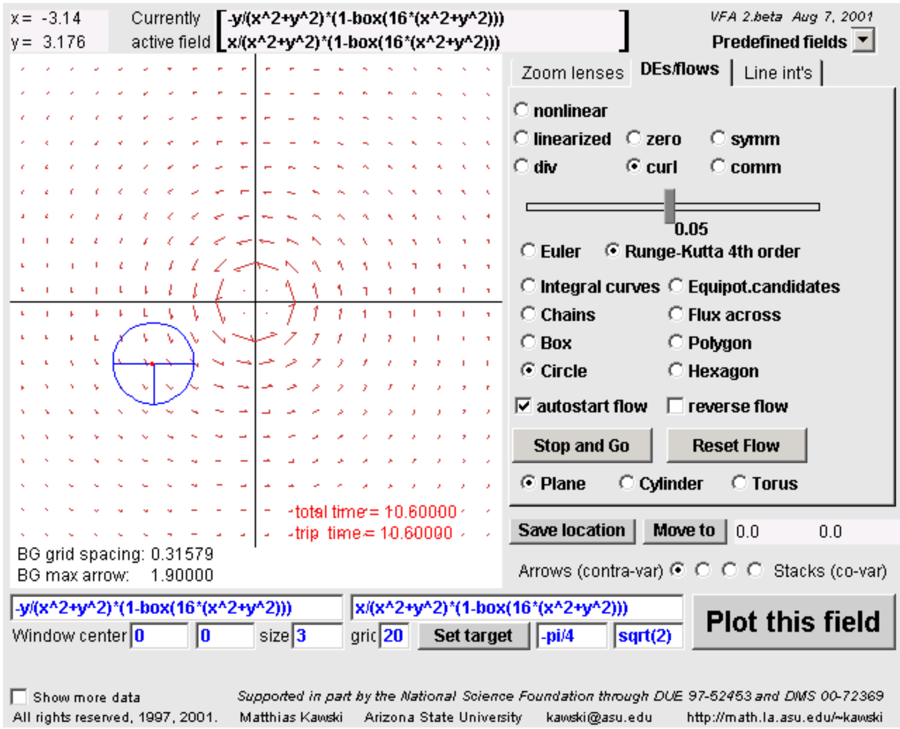

Figure 3: A rigid body floating in the irrotational magnetic field

Arguably the most important exercise is to contrast the animations of the rotation (select **Curl**) of the flow of the harmonic oscillator $F(x,y) = (-y,x)$ and of the *magnetic field* $F(x,y) = (x^2+y^2)^{-1}(-y,x)$. In the linear field a rigid body spins about itself at the same rate as it rotates about the origin – we like to write $\omega = \Omega$ – the animation reminds us of the Moon orbiting about the Earth, never showing us its back side. On the other hand, in the irrotational field the orientation of a rigid body is fixed *relative to* an inertial frame. Such experiments with the VFA II quickly cure the usual misconceptions of students who confuse the infinitesimal notion of irrotational with naive global impressions of "rotating" fields. These investigations readily carry over to other fields that arise from complex analytic functions and which are commonly used to model incompressible, laminar fluid flows. The VFA II provides predefined examples such as the **Fluid flow past a cylinder**.

Further suggested explorations on this flow panel address chaotic behavior and periodic attractors: change the topology to the compact **Torus** and start with the **Symmetric** part of a field.

A different line of investigation takes a *co-variant* point of view, and asks whether a given vector field could be the gradient field of some potential function. The starting point is to generate **Equipot.candidates** families of curves that are everywhere orthogonal to the field, and as such are candidates for equipotential curves. Analyzing their relative spacing compared to the magnitude of the field determines whether they truly represent a contour plot of such potential function.

4 Example: Line integrals and Stokes' theorem

The third panel addresses *the other way* in which vector fields (interpreted as differential forms) may be integrated: over curves, surfaces etc. One typically interprets the result as e.g. the *work done* when travelling along a curve in a force field, or as the total flux across a curve (as a volume/area per time). The VFA II provides for either view – here we shall concentrate on the *flux* view which is somewhat more intuitive to visualize.

The starting point is to consider how the flux of a constant field across a line segment depends on magnitude of the field, the length of the curve and the angle in between them. From here one quickly proceeds to polygonal curves and smooth curves – thereby visually supporting the development of the respective Riemann integrals.

A worthwhile experiment investigates how the total flux of a linear field depends on the location, shape, and size of a closed curve (start with polygons). Most experimenters are surprised to find that the integral is independent of the shape and location of the curve, and that it scales by the (signed) area of the region inside the curve. Using carefully chosen example such as $L(x, y) = (8x - 2y, 5x + 3y)$ it is a great experience to *discover* the multiplier that yields the value of the integral for any given area. Clearly this multiplier only depends on the field, not the curve. The VFA II invites the experimenter to flip back to the derivatives panel to visually confirm the conjectures. Together, this investigation is the basis for Green's theorem in the plane, and a precursor for the divergence theorem and any version of Stokes' theorem. The linear / polygonal version of its proof is most accessible, and indeed does not require any calculus, yet provides deep insight.

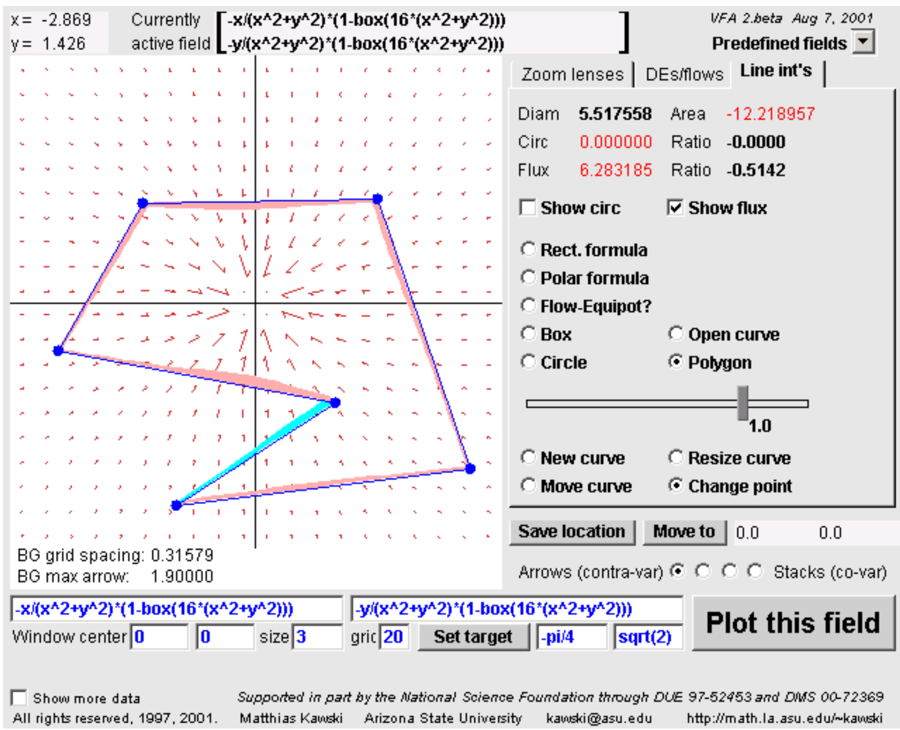

Figure 4: The flux of the gravitational field across a curve and its winding number

As a first teaser towards the general integral theorems investigate the values of either kind of line integral of both the *gravitational/electric field* $G(x, y) = (x^2 + y^2)^{-1}(-x, -y)$ and the *magnetic field* $H(x, y) = (x^2 + y^2)^{-1}(-y, x)$ over any closed curves. Naturally such curves should not pass *through the wire* or *through the Earth*. Using the options **Resize curve**, **Move curve**, and **Change point** the experimenter quickly discovers that the values of the integrals only depends on the *winding number* of the curve about the origin. Moreover, my students are routinely startled when they notice with disbelief that the integral over a triangle (!) (or square, etc.) can yield 2π. These students then demand to see a *proof* of Stokes' theorem that explains what they discovered!

For general nonlinear fields one of the most important experiments investigates the ratio of the line integral and the enclosed area as the curve shrinks into a point. Using the **Save location** and **Move to** buttons it is easy to shrink curves of various shapes into the same point and to discover that the ratio of integral and area indeed has a limit at every point, and that this limit is independent of the shape of the curve. From here it is an easy to obtain elegant arguments that establish the integral theorems of vector calculus.

5 Summary

We presented numerous highly interactive explorations that are destined to support the learning and teaching of vector calculus, together with forming strong linkages to differential equations, linear algebra, and complex analysis. A key feature is a highly visual language that takes the place of the traditional almost complete dominance by an arcane algebraic-symbolic language.

In addition to serving as a practical tool that helps the learning, teaching, and understanding of the special mathematical topic per se, this software tool also shall serve as a model in more general ways. Some notable features are:

- Which *data* are entered by symbolic formula, which are generated dynamically by *drawing* them with the mouse, dragging an object or using a sliding bar? Typical examples are the data that define the dynamics or a force field versus initial conditions or a curve in the space.
- Which outputs are presented as visual images / animations and which are presented numerically. Some, like the *flux* are presented both ways.
- The user should be able to clearly focus on the main item without undue distraction, yet should still be able to discover subtle links to other areas – the pastel colored eigenspaces are a typical example.
- When *things go wrong* – e.g. when looking for a derivative at a discontinuity, the tool should not crash, but provide intuitive forceful feedback.
- The main challenge should be understanding the mathematics, not navigating the software interface.

A key to holding the learners interest and excitement is that the software is *open*, inviting explorations far beyond a single purpose. All too many applets only support very specific *canned* experiments reminiscent of many a chemistry lab class. Our students, even small children have made numerous startling conjectures such as raising the question whether the *dolphin* (children do not draw boxes, they create much more exciting regions!) in a nonlinear flow on a torus always will come back to the original size? The well-prepared teacher immediately recognizes that this involves the integral of the divergence over periodic orbits . . . no matter what the level, the experimenter develops a deep sense of ownership over the conjectures/theorems that (s)he made her/himself. while the teacher is to help steer the experimenter to an age-appropriate *explanation/proof*. Next to the immediate practical utility, and the promotion of a visual language, possibly the most important contribution of the VFA II is as a model for such an open architecture that invites true exploration and discovery.