CDC02-REG 0178

Lack of convexity for tangent cones of needle variations[1]

Rosa-Maria Bianchini
Dipartimento di Matematica "Ulisse Dini"
Università degli Studi di Firenze
Viale Morgagni 67a,
50134 Firenze, Italy
bianchini@unifi.it

Matthias Kawski[2]
Department of Mathematics and Statistics
Arizona State University
Tempe, Arizona 85287-1804, U.S.A.
http://math.asu.edu/~kawski
kawski@asu.edu,

Keywords: Control variations, tangent cones, convexity, nonlinear controllability, optimality. **AMS subject classification**: 93B05, 49K15

Abstract

1 Introduction

A fundamental dichotomy for a control system together with a reference trajectory is whether the trajectory is *optimal*, or whether the system is *controllable* about this trajectory. Topologically, this translates into the reference trajectory lying either on the boundary, or in the interior of the *funnel of reachable sets*. Derivatives of the *endpoint map* with respect to the controls serve as the main analytic tool: If the derivative has full rank, then the trajectory lies in the interior and the system is *controllable*. Conversely, a necessary condition for *optimality* is that the derivative does not have full rank, leading to the Pontryagin Maximum Principle [16] (which is a *first derivative* test).

Along many reference controls the first derivative do not have full rank, and there have been many efforts to obtain higher order conditions for controllability and optimality, with Krener's high-order maximal principle [14], Stefani's high order conditions for optimality [17], and Sussmann's general theorem for controllability [19] a few prominent ones. See [20] for the current state of the art unifying nonsmooth with differential geometric approaches.

Underlying algebraic rank conditions for optimality and controllability in terms of iterated Lie brackets (which generalize the Kalman rank condition to nonlinear systems) are notions of higher order tangent. These are basically high-order directional derivatives of the endpoint map, obtained from *families of control variations*, i.e. *curves* in the space of admissible controls. Open mapping theorems guarantee that if a *cone of tangent vectors* is the whole tangent space, then the reference trajectory lies in the interior, see [9, 10, 15, 16] for some classical statements.

Aside from considering curves such that $\|u_s - u^*\|_1 \searrow 0$ as $s \searrow 0$ (where $u_0 = u^*$ is the reference control), compare [8], optimal control theory has much utilized *"needle variations"*. Loosely speaking, these are such that the variation u_s agrees with the reference control u^* everywhere except on a finite number of intervals whose combined length goes to zero as $s \searrow 0$. The main attraction of such needle variations is that they are conceptually easy to combine for the purpose of generating convex combinations of tangent vectors. Of course, this requires that the intervals on which the variations differ from the reference control, to eventually (for sufficiently small $s > 0$) be disjoint. Alternatively, one might require that every needle variation be *moveable*, i.e. one must be able to *move* the variation by a small time-amount and still be able to generate the same tangent vector. Precise technical specification of such conditions can be involved.

For analytic, affine control systems with a *stationary* reference trajectory the sets of tangent vectors have especially nice convexity and approximation properties compare e.g. [9, 10], allowing one to dispense with subtle technical requirements on the curves $s \mapsto u_s$, and leading to a wealth of necessary and sufficient conditions for controllability and optimality, compare [19]. On the other hand, for nonstationary reference trajectories one needs much more careful notions of variations, tangent objects and open-mapping theorems. A persistent open question has been to which extent such delicate technical conditions [1, 3, 4, 5, 6, 8, 9, 10, 13, 20]. are necessary (beyond facilitating proofs for nice cases). This article provides a carefully constructed sequence of simple systems which show that even for very benign systems, the *usual* conditions that needle variations do not collide (or that they are *moveable* by sufficiently large amounts) are essential for guaranteeing convexity of the tangent objects. This, in turn, is essential for practical applicability to decide optimality. These examples also further raise deep questions [2] about the structural stability of nonlinear controllability properties: They demonstrate that the controllability (or the lack thereof) of nilpotent approximating systems need not reflect the controllability (or the lack thereof) of the original systems. These suggest limita-

[1] This is a condensed announcement of the main results of the full paper [7] which contains all proofs, and which has been submitted to SIAM Journal of Control and Optimization.

[2] supported by NSF-grant DMS 00-72369 and CNR

tions to extending the usual arguments using nilpotent approximations have played a critical role in obtaining many classical controllability and optimality results, e.g. [4, 5, 11, 17, 18, 19].

After this introduction, the article gives and reviews some technical definitions. The third section contains most of the key innovative constructions and estimates. Subsequent sections transfer the results to a system with nonstationary reference trajectory, and analyze perturbations which are higher order, but nonetheless destroy controllability. Only selected outlines of key arguments are given. Refer to the full paper [7] for complete proofs, further results and discussion.

2 Control variations and approximating cones

The systems under consideration are finite-dimensional, deterministic, analytic systems (possibly including an analytic *output* function) that are affine in the control.

$$\dot{x} = f_0(x) + \sum_{i=1}^{\ell} u_i(t) f_i(x),\ x(0) = 0,\ y = \varphi(x). \quad (1)$$

Here $x \in \mathbb{R}^n$, $f_i : \mathbb{R}^n \mapsto T\mathbb{R}^n$ are analytic vector fields, the controls $u : [0,T] \mapsto U \subseteq \mathbb{R}^\ell$ are measurable functions defined on finite intervals and take values in a compact convex set U, usually $U = [-1,1]^\ell$. The output map $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^p$ is analytic. The solution curves of (1) corresponding to a control u, starting from $x(0) = 0$ are denoted by $x(t,u)$, or when no confusion arises, by $x(t)$. Their images under the output map are denoted $y(t,u) = \varphi(x(t,u))$, or $y(t)$. Throughout this article we identify the tangent spaces $T_p\mathbf{R}^n$ with $\mathbf{R}^n$. Open balls always are defined via the 1-norm, i.e. $B_p(\rho) = \{x \in \mathbf{R}^n : \|x-p\| = \max_{i=1}^n \|x_i - p_i\| < \rho\}$.

Definition 2.1 *The reachable sets $\mathcal{R}(t)$ consist of all points $x(t,u)$ reached in time t by trajectories of (1) from $x(0) = 0$ by means of admissible controls.*
Given a reference trajectory $x^*(t) = x(t,u^*)$, the system (1) *is small-time locally controllable* (STLC) *about x^* if $x^*(t) \in \operatorname{int}\mathcal{R}(t)$ for all $t > 0$. The system* (1) *is small-time locally output controllable* (STLOC) *about $y^* = \varphi(x^*)$ if $y^*(t) \in \operatorname{int}(\varphi(\mathcal{R}(t)))$ for all $t > 0$.*

Definition 2.2 *A one-parameter family of control variations of a reference control $u^* : [0,T] \mapsto U$ is a curve $s \mapsto u_s \in \mathcal{U} = \{u : [0,T] \mapsto U$ measurable $\}$ defined for $s \in [0,s_0]$ for some $s_0 > 0$, and with $u_0 \equiv u^*$. Such a family is called a family of control variations* at zero *if $u_s(t) = u^*(t)$ for all $t \in (s,T]$.*

We conveniently identify u_s and its restriction to $[0,s]$. Denote by $\Phi : (t,p) \mapsto \Phi_t(p) = x(t,u^*;p)$ and by $\Phi_{T*} : T_p\mathbf{R}^n \mapsto T_{\Phi_t(p)}\mathbf{R}^n$ the flow corresponding to the reference control u^*, and its tangent map, respectively.

Definition 2.3 *Suppose $u^* : [0,T] \mapsto U$ generates the smooth trajectory $x^* : [0,T] \mapsto \mathbf{R}^n$. A vector $v \in \mathbb{R}^n$ is an m-th order tangent vector to the family of reachable sets $\{\mathcal{R}(T)\}_{T\geq 0}$ at zero, written $v \in \mathcal{K}_0^m$, if there exists a family of control variations $u_s : [0,s] \mapsto U$ such that*

$$x(s,u_s) = x(s,u^*) + s^m \Phi_{s*,0}(v) + o(s^m) \quad (2)$$

A vector $v \in \mathbb{R}^p$ is an m-th order tangent vector to the output-reachable sets of (1) *with stationary reference trajectory $x^* \equiv 0$, written $v \in \mathcal{K}_\varphi^m$, if there exists a family of control variations $u_s : [0,s] \mapsto U$ such that*

$$\varphi(x(s,u_s)) = \varphi(x(0)) + s^m v + o(s^m) \quad (3)$$

A vector $v \in \mathbb{R}^n$ is an m-th order tangent vector to the reachable set $\mathcal{R}(T)$ at $x^*(T)$, written $v \in \mathcal{K}_T^m$, if there exists a family of control variations $u_s : [0,T] \mapsto U$ of u^* such that (*writing $\Delta u = u_s - u^*$*)

$$x(T,u_s) = x(T,u^*) + \|\Delta u\|_1^m \Phi_{T*,0}(v) + o(\|\Delta u\|_1^m). \quad (4)$$

Write $\overline{\mathcal{K}}_T^m$, $\overline{\mathcal{K}}_0^m$, and $\overline{\mathcal{K}}_\varphi^m$ for the cones $\overline{\mathcal{K}}_0^m = \{\lambda v : v \in \mathcal{K}_0^m,\ \lambda \geq 0\}$ etc. generated by $\mathcal{K}_T^m$, $\mathcal{K}_0^m$, and $\mathcal{K}_\varphi^m$, respectively. Note, that in this setting no regularity whatsoever is assumed or required for the maps $s \mapsto u_s$.

To render sets of tangent vectors useful as *"approximating cones"* for obtaining optimality results via classical open mapping theorems [15, 16], one needs to establish their convexity using *continuous* multi-parameter families of control variations. Given n curves $s \mapsto u_s^{(i)} \in \mathcal{U}$, that generate the tangent vectors $v^{(i)} \in \mathcal{K}_0^m$, $i = 1, \ldots n$, in an effort to generate convex combinations $(c_1 v_1 + \ldots + c_n v_n)$, it is natural to utilize convex combinations of (suitable reparameterizations) of the original curves

$$s \mapsto u_{s,c_1,\ldots c_n} = c_1 u^{(1)}_{\alpha_1(c,s)} + \ldots + c_n u^{(n)}_{\alpha_n(c,s)}. \quad (5)$$

Variations that lend themselves especially well to generating such convex combinations are needle variations:

Definition 2.4 *A family of control variations $s \mapsto u_s : [0,T] \mapsto U$ defined for $s \in [0,s_0]$ with $s_0 > 0$ is a (family of)* needle variations *of the reference control $u^* = u_0$ if there exist a constant $C > 0$ and a finite number of pairs of increasing functions $s \mapsto a_s^{(k)}$ and decreasing functions $s \mapsto b_s^{(k)}$ defining intervals $[a_s^{(k)}, b_s^{(k)}] \subseteq [0,T]$, $k = 1, \ldots N$, such that*

$$\operatorname{supp}(u_s - u^*) \subseteq \bigcup_{k=1}^{N} [a_s^{(k)}, b_s^{(k)}], \quad \text{and} \quad (6)$$

$$\sum_{k=1}^{N} (b_s^{(k)} - a_s^{(k)}) \leq Cs \quad \text{for all } s \leq s_0 \quad (7)$$

Two such families of variations are easily combined unless the intervals on which they disagree from u^* overlap for all small $s > 0$. In that case, one might be tempted to simply *move* the interval in one of the families. The continuous dependence of solution curves on the data suggests that after such small (or vanishing, as $s \searrow 0$)

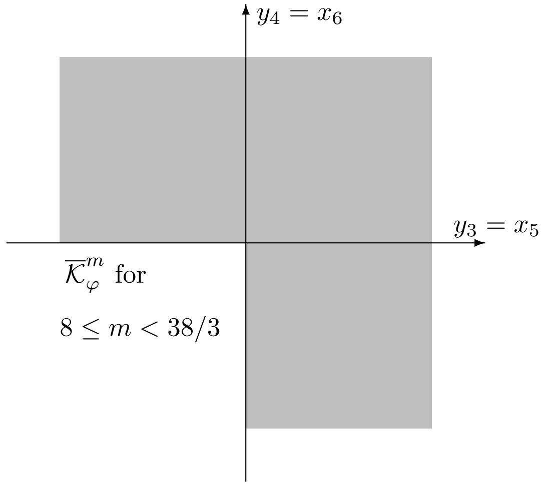

Figure 1: Cross-section of approximating cone for (8)

moves, the combined variations might still generate the desired tangent vectors – this article shows that this need not be the case, even for very benign systems!

Note that a family of variations *at zero* as defined earlier automatically qualifies as a family of needle variations as one may take $N = 1$ and intervals defined by $a_s^{(1)} = 0$ and $b_s^{(1)} = s$. Definition 2.4 also includes other common constructions with $N = 1$ and $[a_s^{(1)}, b_s^{(1)}] = [\alpha, \alpha + s]$ (*right* variations at $t = \alpha$), or $[a_s^{(1)}, b_s^{(1)}] = [\beta - s, \beta]$ (*left* variations at $t = \beta$). The constructions in this article use mainly (right) needle variations at zero. But the main claim of *"needle variations that cannot be summed"* holds even if their *combinations* are allowed to lie in the larger class of definition 2.4.

3 A "barely" output-controllable system

This section analyzes the images of the reachable set of an affine control system under the output map φ.

$$\begin{array}{rclr} \dot{x}_1 &=& u_1 & \varphi(x) = (x_1, x_2, x_5, x_6) \\ \dot{x}_2 &=& u_2 & x(0) = 0 \\ \dot{x}_3 &=& x_1^2 & |u_1(\cdot)| \le 1 \\ \dot{x}_4 &=& x_2^2 & |u_2(\cdot)| \le 1 \\ \dot{x}_5 &=& x_4 x_1^2 - x_1^7 & \\ \dot{x}_6 &=& x_3 x_2^2 - x_2^7 & \end{array} \tag{8}$$

Figure 1 pictorially summarizes key properties of this system, showing cross-sections of approximating cones of the image $\varphi(\mathcal{R}(t))$.

Let $K_{00+*} = \{y \in \mathbb{R}^4 \colon y_1 = y_2 = 0,\ y_3 \ge 0\}$ and $K_{00*+} = \{y \in \mathbb{R}^4 \colon y_1 = y_2 = 0, y_4 \ge 0\}$ denote half-subspaces of the output space. Using bang-bang controls with at most two nonzero *pieces* one easily, explicitly constructs controls steering the system to any given output inside K_{00+*} and K_{00*+} in the expected time:

Theorem 3.1 *There exists a positive constant $C > 0$ such that if $T > 0$ is sufficiently small, the image of the reachable set contains the intersection of the union of the half-spaces with a ball growing with T^8,*

$$\varphi(\mathcal{R}(t)) \supseteq B_0(CT^8) \cap (K_{00+*} \cup K_{00*+}) \tag{9}$$

On the other hand, a delicate calculation outlined below, shows that points in the remaining quadrant $K_{00--} = \{y \in \mathbb{R}^4 \colon y_1 = y_2 = 0, y_3 < 0, y_4 < 0\}$ can, if at all, only be reached much more slowly:

Theorem 3.2 *Suppose $0 \le T < 1$, $x_1(T, u) = 0$, and $x_2(T, u) = 0$. There exists a constant $C > 0$ such that if $x_5(T, u) < 0$ then $x_6(T, u) > -CT^{38/3}$, and if $x_6(T, u) < 0$ then $x_5(T, u) > -CT^{38/3}$.*

Outline of key steps in the proof. A naive effort to reach points with images in K_{00--} might start with controls u_1 and u_2 with disjoint support. First u_1 is used (with $u_2 \equiv 0$, and thus also $x_4 \equiv 0$) to reach points of the form $(0, 0, q_3, 0, -q_5, 0)$ with $q_3, q_5 > 0$. Then use u_2 (while keeping $u_1 \equiv 0$) to reach points $(0, 0, q_3, q_4, -q_5, -q_6)$ with all $q_i > 0$ using a suitable control u_2.

But this second step requires that *sufficient* time is available, depending on the size of $|q_3|$, as a comparison with the simple system shows:

$$\begin{array}{rclrcl} \dot{x}_1 &=& u & |u(\cdot)| &\le& 1 \\ \dot{x}_2 &=& c x_1^2 - x_1^7 & x(0) &=& 0 \end{array} \tag{10}$$

For $c \neq 0$ this system (10) is clearly not STLC about $x = 0$. However, if (and only if) $T > 2\left(\frac{8}{3}|c|\right)^{\frac{1}{5}}$ then $0 \in \operatorname{int}\mathcal{R}(T)$.

The first key step in the calculation is to *abstract* the notion of which direction is being generated first (as in general u_i are only assumed to be measurable). Since

$$\int_0^T x_2^2(t) \int_0^t x_1^2(s)\, ds\, dt + \int_0^T x_1^2(t) \int_0^t x_2^2(s)\, ds\, dt = \tag{11}$$

$$= \int_0^T x_1^2(t)\, dt \int_0^T x_2^2(t)\, dt$$

we may assume without loss of generality that

$$\int_0^T x_2^2(t) \int_0^t x_1^2(s)\, ds\, dt \ge \tfrac{1}{2} \int_0^T x_2^2(t)\, dt \int_0^T x_1^2(t)\, dt, \tag{12}$$

whose right side we rewrite as an iterated integral

$$\int_0^T x_2^2(t) \int_0^t x_1^2(s)\, ds\, dt \ge \tfrac{1}{2} \int_0^T x_2^2(t) \int_0^T x_1^2(s)\, ds\, dt \tag{13}$$

Now the *"weight"* $\int_0^T x_1^2(s)\, ds$ against which x_2^2 is integrated is constant, facilitating comparison with (10). Assume $x_6(T) < 0$, then

$$0 > x_6(T) = \int_0^T (x_3^2(t) x_2^2(t) - x_2^7(t))\, dt \ \ge \tag{14}$$

$$\geq \tfrac{1}{2}\int_0^T x_2^2(t)\,(\int_0^T x_1^2(s)\,ds - 2x_2^5(t))\,dt$$

and thus

$$\xi_2 \overset{\text{def}}{=} \max_{0\leq t\leq T} x_2(t) > \left(\tfrac{1}{2}\int_0^T x_1^2(s)\,ds\right)^{\frac{1}{5}}. \qquad (15)$$

The term on the right hand side may be considered an *energy*, that is required to move $x_6(0) = 0$ to $x_6(T) < 0$. The next step is to find a lower bound for this energy in terms of the displacement. Using that $x_1(0) = x_1(T) = 0$ and $|\dot{x}_1(\cdot)| \leq 1$ gives the crude estimate

$$\int_0^T x_1^2(s)\,ds \geq 2\cdot\left(\tfrac{1}{3}\right)\cdot\xi_1^3 \ \overset{\text{def}}{=}\ \tfrac{2}{3}\max_{0\leq t\leq T} x_1(t) \qquad (16)$$

On the other hand, Minkowski's inequality yields

$$-x_5(T) = \int_0^T (x_1^7(t) - x_4(t)x_1^2(t))dt \leq \qquad (17)$$

$$\leq \int_0^T |x_1^7(t)|\,dt \ \leq T\ \xi_1^7$$

Combined with (15), the two latter estimates (16) and (17) result in

$$\xi_2 > \left(\tfrac{1}{2}\int_0^T x_1^2(s)\,ds\right)^{\frac{1}{5}} \geq \left(\tfrac{1}{3}\,\xi_1^3\right)^{\frac{1}{5}} \geq \left(\tfrac{1}{3}\left(\frac{-x_5(T)}{T}\right)^{\frac{3}{7}}\right)^{\frac{1}{5}} \qquad (18)$$

Using that $\xi_2 \leq \frac{1}{2}T$ (from $u_2(\cdot)| \leq 1$ together with $x_2(0) = x_2(T) = 0$) this implies (for some constant $C > 0$)

$$-x_5(T) \leq 3^{\frac{1}{5}}\cdot\xi_2^{\frac{35}{3}}\cdot T \leq CT^{\frac{38}{3}}. \qquad \blacksquare \qquad (19)$$

Theorem 3.3 *There exist a constant $C > 0$ such that for all sufficiently small $T > 0$ the image $\varphi(\mathcal{R}(T))$ of the reachable set of system* (8) *contains the open ball $B_0^\infty(CT^{40/3})$. and thus the system* (8) *is small-time locally output controllable* (STLOC) *about $y = 0$.*

These statements concerning the images of the reachable sets translate into the following for the approximating tangent objects:

Theorem 3.4 *For $8 < m < 38/3$ and system* (8), *$(0,0,a,b) \in \overline{\mathcal{K}}_\varphi^8$ if and only if $a \geq 0$ or $b \geq 0$, (and the approximating cones $\overline{\mathcal{K}}_\varphi^8$ of the output-reachable set are not convex). For $m \geq 40/3$ the approximating cone $\overline{\mathcal{K}}_\varphi^m$ is the whole tangent space $T_0\mathbb{R}^4$.*

In particular, the tangent vectors $(0,0,-1,0),(0,0,0,-1) \in \overline{\mathcal{K}}_\varphi^8$ are generated by *needle-variations* of the zero reference control, but their convex combination $(0,0,-\frac{1}{2},-\frac{1}{2})$ cannot be generated as a tangent vector of order less than (38/3) by any family of control variations, and thus, a fortiori, not by any family of needle variations.

Main elements of the construction. A main task of generating any tangent vectors in K_{00--} is to suitably combine the families of control variations

$$u_s^{(1)}(t) = \begin{cases} (+1,0) & \text{if } 0 \leq t < \frac{s}{2} \\ (-1,0) & \text{if } \frac{s}{2} \leq t < s \end{cases} \qquad (20)$$

and

$$u_s^{(2)}(t) = \begin{cases} (0,+1) & \text{if } 0 \leq t < \frac{s}{2} \\ (0,-1) & \text{if } \frac{s}{2} \leq t < s \end{cases} \qquad (21)$$

which steer to points $y(s,u_s^{(1)}) = (0,0,-2^{-7}s^8,0)$ and $y(s,u_s^{(2)}) = (0,0,0,-2^{-7}s^8)$ on the negative y_3 and y_4 axes, to obtain families of control variations that steer to points in the third quadrant $y_3 < 0$ and $y_4 < 0$ in the plane $y_1 = y_2 = 0$. Since these are needle variations *at the same point* $t = 0$, the standard approach is to translate one family by a small amount (the length of the other control), and count on *"continuous dependence on initial conditions"* (i.e. basically Gronwall's inequality). However, it is not hard to see that no such translation of linearly rescaled families of control variations can do the job. The key innovation here is to rescale the first family of needle variations by a power function $s \mapsto s^r$ for some exponent $r > 1$. As a result, the second variations which themselves require intervals of order s will only be *shifted* by a time of order s^r. In this specific case, it can be shown that the critical value is $r = \frac{5}{3}$, and the following family of needle variations will indeed generate tangent vectors in K_{00--}, albeit at the much higher order $m = \frac{40}{3}$.

$$u_s(t) = \begin{cases} (+1,0) & \text{if } \ 0 \leq t < t_1 = \frac{1}{2}c_1 s^{5/3} \\ (-1,0) & \text{if } \ t_1 \leq t < t_2 = c_1 s^{5/3} \\ (0,+1) & \text{if } \ t_2 \leq t < t_3 = c_1 s^{5/3} + \frac{1}{2}c_2 s \\ (0,-1) & \text{if } \ t_3 \leq t < T = c_1 s^{5/3} + c_2 s. \end{cases} \qquad (22) \quad \blacksquare$$

4 A nonstationary reference trajectory

This section demonstrates how to transfer the above results about *output controllability* to *controllability about* a (nonstationary) *reference trajectory*, the classical setting to which the original open questions referred. We state the corresponding findings about the reachable set and approximating cones, and comment on the *strategy* that makes this construction *work*. This section doe not require any major innovative calculations or estimates.

$$\begin{array}{lcll} \dot{x}_1 & = & u_1 & x^*(t) = (0,0,t,t,0,0) \\ \dot{x}_2 & = & u_2 & x(0) = 0 \\ \dot{x}_3 & = & x_1^2 + (1+u_{01}) & |u_{01}(\cdot)| \leq 1 \\ \dot{x}_4 & = & x_2^2 + (1+u_{02}) & |u_{02}(\cdot)| \leq 1 \\ \dot{x}_5 & = & x_4x_1^2 - x_1^7 & |u_1(\cdot)| \leq 1 \\ \dot{x}_6 & = & x_3x_2^2 - x_2^7 & |u_2(\cdot)| \leq 1 \end{array} \qquad (23)$$

The key in this construction is that the *lower order* (i.e. apparently dominant) positive definite components $\int_0^T x_1^2(t)dt$ and $\int_0^T x_2^2(t)dt$ are effectively aligned with the direction of the nonstationary reference trajectory, and that the *zero speed* $\|\dot{x}\| = 0$ is on the *boundary* of the set of admissible *velocities* (on the hyperplane $x_1 = x_2 = 0$ which includes the reference trajectory).

With the fixed boundary velocity $u_{01} = u_{02} = -1$ the system may be considered a two-input system (with controls (u_1, u_2)) about the *stationary reference trajectory* $x \equiv 0$ (basically system (8)) which is not STLC (as obviously it is impossible to reach points x with $x_3 < 0$ or $x_4 < 0$ from $x = 0$). But in (23) these uncontrollable directions are aligned with (as opposed to transversal to) the nonstationary reference trajectory, and thus controllability involves comparison with $\dot{x}_3 \equiv \dot{x}_4 \equiv 1$ (as opposed to $\dot{x}_3 \equiv \dot{x}_4 \equiv 0$).

Theorem 4.1 *For system* (23) *and* $m \geq \frac{40}{3}$ *the cone* $\overline{\mathcal{K}}_0^m$ *is the whole tangent space* $T_0\mathbb{R}^6$*, but for* $8 \leq m < \frac{38}{3}$ *the cones* $\overline{\mathcal{K}}_0^m$ *are not convex as*

$$(0,0,0,0,v_5,v_6) \in \overline{\mathcal{K}}_0^m \iff (\, v_5 \geq 0 \textbf{ or } v_6 \geq 0\,)\,. \qquad (24)$$

Theorem 4.2 *There exists* $C > 0$ *such that for every* $T > 0$ *sufficiently small, the reachable set* $\mathcal{R}(T)$ *of system* (23) *contains the open ball* $B^{\infty}_{x^*(T)}(CT^{40/3})$ *centered at* $x^*(T)$ *and thus* (23) *is STLC about the reference trajectory* $x^*(t) = (0,0,t,t,0,0)$*.*

5 Loss of controllability under perturbations

The delicate constructions of convex combinations of tangent vectors, but *"only at higher orders"*, have important ramifications regarding the robustness of the controllability properties. Whereas for the original systems (8) (23) it was possible to generate certain convex combinations of 8^{th} order tangent vectors as tangent vectors of order $\frac{40}{3}$ (but not of order less than $\frac{38}{3}$), very small additional *perturbations* may destroy this property, and the controllability!

$$\begin{array}{rcll}
\dot{z}_1 & = & u_1 & \varphi(z) = (z_1, z_2, z_5, z_6) \\
\dot{z}_2 & = & u_2 & z(0) = 0 \\
\dot{z}_3 & = & z_1^2 & |u_1(\cdot)| \leq 1 \\
\dot{z}_4 & = & z_2^2 & |u_2(\cdot)| \leq 1 \\
\dot{z}_5 & = & z_4 z_1^2 - z_1^7 + z_1^{10} + z_2^{10} & \\
\dot{z}_6 & = & z_3 z_2^2 - z_2^7 + z_1^{10} + z_2^{10} &
\end{array} \qquad (25)$$

Theorem 5.1 *For* $0 \leq T < 1$ *the image* $\varphi(\mathcal{R}(T))$ *of the reachable set of system* (25) *does not contain any points of the form* $(0,0,\eta_3,\eta_4)$ *with* $\eta_3 < 0$ *and* $\eta_4 < 0$*. and thus* (25) *is not small-time locally output controllable* (STLOC) *about* $y = 0$*.*

Theorem 5.2 *For any* $m \geq 8$ *and system* (25)*,* $(0,0,a,b) \in \overline{\mathcal{K}}_\varphi^m$ *if and only if* $a \geq 0$ *or* $b \geq 0$*. In particular,* $(0,0,-1,0) \in \overline{\mathcal{K}}_\varphi^8$ *and* $(0,0,0,-1) \in \overline{\mathcal{K}}_\varphi^8$ *are tangent vectors generated by needle variations, but their convex combination* $(0,0,-\frac{1}{2},-\frac{1}{2}) \notin \overline{\mathcal{K}}_\varphi^m$ *is not a tangent vector of any order.*

Proof. Write z and x for corresponding solutions of the perturbed and unperturbed systems (8) and (25), e.g.

$$z_5(t,u) = x_5(t,u) + \int_0^t \left(x_1^{10}(\tau,u) + x_2^{10}(\tau,u)\right) d\tau \quad (26)$$

Assuming $x_6(T) < 0$, use estimates analogous to (16) and (18), and conclude

$$\int_0^T x_2^{10}(t)\, dt \geq \tfrac{2}{11}\, \xi_2^{11} \geq \qquad (27)$$
$$\geq \tfrac{2}{11} \left(\left(\frac{-x_5(T)}{T} \right)^{\frac{3}{35}} \right)^{11} \geq |x_5(T)| > 0.$$

If $T < 1$, clearly also $|x_5(T)| < 1$, which shows that $z_5(T) = x_5(T) + \int_0^T |x_2(t)|^{10} dt \geq 0$ if $z_6(T) < 0$. ■

For corresponding results for loss of controllability about a nonstationary reference trajectory consider

$$\begin{array}{rcll}
\dot{z}_1 & = & u_1 & z^*(t) = (\,0,0,t,t,0,0) \\
\dot{z}_2 & = & u_2 & z(0) = 0 \\
\dot{z}_3 & = & z_1^2 + (1 + u_{01}) & |u_1(\cdot)| \leq 1 \\
\dot{z}_4 & = & z_2^2 + (1 + u_{02}) & |u_2(\cdot)| \leq 1 \\
\dot{z}_5 & = & z_4 z_1^2 - z_1^7 + z_1^{10} + z_2^{10} & |u_{01}(\cdot)| \leq 1 \\
\dot{z}_6 & = & z_3 z_2^2 - z_2^7 + z_1^{10} + z_2^{10} & |u_{02}(\cdot)| \leq 1
\end{array} \qquad (28)$$

Theorem 5.3 *For* $0 \leq T < 1$ *the reachable set* $\mathcal{R}(T)$ *of system* (28) *does not contain any points of the form* $(0,0,0,0,q_5,q_6)$ *with* $q_5 < 0$ *and* $q_6 < 0$*, and thus* (28) *is not* STLC *about* $z^*(t) = (0,0,t,t,0,0)$*.*

Theorem 5.4 *For any* $m \geq 8$ *and system* (28)*,* $(0,0,0,0,a,b) \in \overline{\mathcal{K}}_0^m$ *if and only if* $a \geq 0$ *or* $b \geq 0$*, and thus the cones* $\overline{\mathcal{K}}_T^m$ *are not convex. In particular,* $(0,0,0,0,-1,0) \in \overline{\mathcal{K}}_T^8$ *and* $(0,0,0,0,0,-1) \in \overline{\mathcal{K}}_T^8$ *are generated by needle variations at zero, but for all* $m > 0$ *and* $T \geq 0$*,* $(0,0,0,0,-\frac{1}{2},-\frac{1}{2}) \notin \overline{\mathcal{K}}_T^m$

6 Conclusion and further outlook

Summarizing, the analysis of the custom-designed systems has yielded two major insights.
The immediate result is a counterexample for a longstanding problem concerning needle variations: The most general notions of tangent vectors do not yield convex cones for neither nonstationary reference trajectories nor for systems with output. Narrower notions that yield convex cones, e.g. by requiring explicitly that needle variations must be *movable* by amounts of the same order as the length of the variation, might fail to detect controllability of some systems.
On a deeper level these examples cast further doubts

on the structural stability and robustness of nonlinear controllability. It is well-known that Taylor approximations do not need preserve controllability, e.g. [3, 18]. However, many controllability results rely on nilpotent approximating systems [4, 11] which in turn are based on *graded structures* associated to families of dilations $\Delta_s(x) = (s^{r_1}x_1, \ldots, s^{r_n}x_n)$, with local coordinates x_i and exponents r_i determined from the ranks of the derived distributions of the Lie algebra $L(f_0, f_1, \ldots f_\ell)$. One algorithmically *approximates* a system (1) by a *nilpotent* system of the same form by replacing each vector field f_i by its *principal part* in the expansion of f_i into a graded series relative to Δ. Roughly speaking, the fundamental result is that if such Δ-*homogeneous* nilpotent approximating system is STLC, then the original system is *STLC* [4, 18, 19]. However, there exist polynomial systems that are STLC, but for which any such homogeneous nilpotent approximating system is not STLC [12]. Similarly, the standard nilpotent approximations of systems (8) and (23) (which are obtained by deleting the x_1^7 and x_2^7 terms) are not STLOC and STLC, respectively, even though the *original* systems are. On the other hand, one may consider (8) and (23) themselves nilpotent approximations (although as nonhomogeneous systems they do not arise from standard algorithms) of the perturbed systems (25) and (28) – but now the approximations are controllable whereas the *original* systems are not. One naturally wonders whether one may construct infinite chains of such approximations or perturbations with alternating controllability properties – which is closely related to the major open problem whether STLC (of analytic systems) is *finitely determined* [2].

Acknowledgements

The second author gratefully acknowledges the hospitality of the Universita degli Studi in Firenze and the support by the Italian CNR for this work.

References

[1] A. Agrachev, *Newton diagrams and tangent cones to attainable sets*, Prog. Systems Control Theory **8** (1991) pp. 11-20.

[2] A. Agrachev, *Is it possible to recognize local controllability in a finite number of differentiations?*, in: Open Problems in Mathematical Systems and Control Theory (1999). (Springer)

[3] R. Bianchini and G. Stefani, *Normal local controllability of order one*, Int. J. Cntrl.**39** (1984) pp.701–714.

[4] R. Bianchini and G. Stefani, *Graded approximations and controllability about a reference trajectory*, SIAM J. Control Optim. **28** (1990) pp. 903-924.

[5] R. Bianchini and G. Stefani, *Local controllability about a reference trajectory: a variational approach*, SIAM J. Control Optim. **31** (1993) pp. 900-927.

[6] R. Bianchini, *Good needle-like variations* Proc. Symposia Pure Math., **64** (1999) pp. 91-101.

[7] R. Bianchini and M. Kawski, *Needle variations that cannot be summed*, SIAM J. Control Optim. (2002) (submitted). (Electronic preprint available on-line at `http://math.asu.edu/~kawski`.)

[8] A. Bressan, *A high-order test for optimality of bang-bang controls* SIAM J. Control Optim. **23** (1985) pp. 38–48.

[9] H. Frankowska, *An open mapping principle for set-valued maps*, J. Math. Anal. Appl. **127** (1987) pp. 172–180.

[10] H. Frankowska, *Local controllability of control systems with feedback*, J. Optim. Theory Appl. **60** (1989) pp. 277–296.

[11] H. Hermes, *Nilpotent and high-order approximations of vector field systems*, SIAM Review **33** (1991) pp. 238–264.

[12] M. Kawski, *Control variations with an increasing number of switchings*, Bull. AMS **18** (1988) pp. 149-152.

[13] H. Knobloch, *Higher Order Necessary Conditions in Optimal Control Theory*, Lect. Notes in Control and Information Sciences **34** (1981). (Springer)

[14] A. Krener, *The high order maximal principle and its application to singular extremals*, SIAM J. Control Optim. **15** (1977) pp. 256–293.

[15] E. Lee and L. Markus, *Foundations of optimal control theory*, (1967) New York. (Wiley)

[16] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mischenko, *The mathematical theory of optimal processes*, (1962) New York. (Wiley)

[17] G. Stefani, *Polynomial approximations to control systems and local controllability*, Proc. 25$^{\text{th}}$ IEEE CDC (1985) pp. 33–38.

[18] H. Sussmann, *Lie brackets and local controllability: a sufficient condition for scalar-input systems*, SIAM J. Control Optim., **21** (1983) pp.686–713.

[19] H. Sussmann, *A general theorem on local controllability*, SIAM J. Cntrl.Optim. **25** (1987) pp. 158-194.

[20] H. Sussmann, *Needle variations and almost lower semicontinuous differential inclusions*, Set-valued Anal. (to appear)