Chapter 1

Bases for Lie algebras and a continuous CBH formula

Matthias Kawski[1]
Department of Mathematics and Statistics
Arizona State University,
Tempe, Arizona 85287-1804, U.S.A.,
http://math.asu.edu/~ kawski,
kawski@asu.edu

1.1 Description of the problem

Many time-varying linear systems $\dot{x} = F(t,x)$ naturally split into time-invariant geometric components and time-dependent parameters. A special case are non-linear control systems that are affine in the control u, and specified by analytic vector fields on a manifold M^n

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} u_k\, f_k(x). \tag{1.1}$$

It is natural to search for solution formulas for $x(t) = x(t,u)$ that, separate the time-dependent contributions of the controls u from the invariant, geometric role of the vector fields f_k. Ideally, one may be able to a-priori obtain closed-form expressions for the flows of certain vector fields. The quadratures of the control might be done in real-time, or the integrals of the controls may be considered new variables for theoretical purposes such as path-planning or tracking.

To make this scheme work, one needs *simple* formulas for assembling these pieces to obtain the solution curve $x(t,u)$. Such formulas are nontrivial since in general the vector fields f_k do not commute: already in the case of linear

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systems, $\exp(sA) \cdot \exp(tB) \neq \exp(sA + tB)$ (for matrices A and B). Thus the desired formulas not only involve the flows of the system vector fields f_i, but also the flows of their iterated commutators $[f_i, f_j]$, $[[f_i, f_j], f_k]$, and so on.

Using Hall-Viennot bases $\mathcal{H}$ for the free Lie algebra generated by m indeterminates $X_1, \ldots X_m$, Sussmann [22] gave a general formula as a directed infinite product of exponentials

$$x(T,u) = \prod_{H \in \mathcal{H}}^{\rightarrow} \exp(\xi_H(T,u) \cdot f_H). \tag{1.2}$$

Here the vector field f_H is the image of the formal bracket H under the canonical Lie algebra homomorphism that maps X_i to f_i. Using the *chronological product* $(U * V)(t) = \int_0^T U(s)V'(s)\,ds$, the iterated integrals ξ_H are defined recursively by $\xi_{X_k}(T,u) = \int_0^T u_k(t)dt$ and

$$\xi_{HK} = \xi_H * \xi_K \tag{1.3}$$

if H, K, HK are Hall words and the *left factor* of K is not equal to H [9, 22]. (In the case of repeated left factors, the formula contains an additional factorial.)

An alternative to such infinite exponential product (in Lie group language: *"coordinates of the* 2^{nd} *kind"*) is a single exponential of an infinite Lie series (*"coordinates of the* 1^{st} *kind"*).

$$x(T,u) = \exp\Big(\sum_{B \in \mathcal{B}} \zeta_B(T,u) \cdot f_B\Big) \tag{1.4}$$

It is straightforward to obtain explicit formulas for ζ_B for some spanning sets $\mathcal{B}$ of the free Lie algebra [22], but much preferable are series that use *bases* $\mathcal{B}$, and which, in addition, yield as simple formulas for ζ_B as (1.3) does for ξ_H.

Problem 1. Construct bases $\mathcal{B} = \{B_k : k \geq 0\}$ for the *free* Lie algebra on a finite number of generators $X_1, \ldots X_m$ such that the corresponding iterated integral functionals ζ_B defined by (1.4) have *simple formulas* (similar to (1.3)), suitable for control applications (both analysis and design).

The formulae (1.4) and (1.2) arise from the *"free control system"* on the free associative algebra on m generators. Its universality means that its solutions map to solutions of specific systems (1.1) on M^n via the evaluation homomorphism $X_i \mapsto f_i$. However, the resulting formulas contain many redundant terms since the vector fields f_B are not linearly independent.

Problem 2. Provide an algorithm that generates for any finite collection of analytic vector fields $\mathcal{F} = \{f_1, \ldots, f_m\}$ on M^n a basis for $L(f_1, \ldots, f_m)$ together with effective formulas for associated iterated integral functionals.

Without loss of generality one may assume that the collection $\mathcal{F}$ satisfies the Lie algebra rank condition. i.e. $L(f_1, \ldots, f_m)(p) = T_pM$ at a specified initial point p. It makes sense to first consider special classes of systems $\mathcal{F}$, e.g. which are such that $L(f_1, \ldots, f_m)$ is finite, nilpotent, solvable, etc. The words *simple* and *effective* are not used in a technical sense in problems 1 and 2 (as in formal studies of computational complexity), but instead refer to comparison with the elegant formula (1.3), which has proven convenient for theoretical studies, numerical computation, and practical implementations.

1.2 Motivation and history of the problem

Series expansions of solution to differential equations have a long history. Elementary Picard iteration of the *universal control system* $\dot{S} = \sum_{i=1}^{m} X_i u_i$ on the free associative algebra over $\{X_1, \ldots, X_m\}$ yields the Chen Fliess series [5, 11, 21]. Other major tools are Volterra series, and the Magnus expansion [14], which groups the terms in a different way than the Fliess series. The main drawback of the Fliess series is that (unlike its exponential product expansion (1.2)) no finite truncation is the exact solution of any *approximating system.* A key innovation is the *chronological calculus* of 1970s Agrachev and Gamkrelidze [1]. However, it is generally not formulated using explicit bases.

The series and product expansions have manifold uses in control beyond simple computation of integral curves, and analysis of reachable sets (which includes controllability and optimality). These include state-space realizations of systems given in input-output operator form [8, 20], output tracking and path-planning. For the latter, express the target or reference trajectory in terms of the ξ or ζ, now considered as *coordinates* of a suitably lifted system (e.g. free nilpotent) and *invert* the restriction of the map $u \mapsto \{\xi_B : B \in \mathcal{B}_N\}$ or $u \mapsto \{\zeta_B : B \in \mathcal{B}_N\}$ (for some finite subbasis $\mathcal{B}_N$) to a finitely parameterized family of controls u, e.g. piecewise polynomial [7] or trigonometric polynomial [12, 17].

The Campbell-Baker-Hausdorff formula [18] is a classic tool to combine products of exponentials into a single exponential $e^a e^b = e^{H(a,b)}$ where $H(a,b) = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] - \frac{1}{12}[b,[a,b] + \ldots$. It has been extensively used for designing piecewise constant control variations that generate high order tangent vectors to reachable sets, e.g. for deriving conditions for optimality. However, repeated use of the formula quickly leads to unwieldily expressions. The expansion (1.2) is the natural *continuous* analogue of the CBH formula – and the problem is to find the most useful form.

The uses of these expansions (1.2) and (1.4) extend far beyond control, as they apply to any dynamical systems that split into different interacting components. In particular, closely related techniques have recently found much attention in numerical analysis. This started with a search for Runge-Kutta-like integration schemes such that the approximate solutions inherently satisfy algebraic constraints (e.g. conservation laws) imposed on the dynamical system [3]. Much effort has been devoted to *optimize* such schemes, in particular minimizing the number of costly function evaluations [16]. For a recent survey see [6]. Clearly the form (1.4) is most attractive as it requires the evaluation of only a single (*computationally costly*) exponential.

The general area of noncommuting formal power series admits both dynamical systems / analytic and purely algebraic / combinatorial approaches. Algebraically, underlying the expansions (1.2) and (1.4) is the Chen series [2], which is well known to be an exponential Lie series, compare [18], thus guaranteeing the existence of the alternative expansions

$$\sum_{w \in Z^*} w \otimes w \stackrel{!}{=} \exp\Big(\sum_{B \in \mathcal{B}} \zeta_B \otimes B \Big) \stackrel{!}{=} \overrightarrow{\prod_{B \in \mathcal{B}}} \exp\left(\xi_B \otimes B\right) \tag{1.5}$$

The first bases for free Lie algebras build on Hall's work in the 1930s on commutator groups. While several *other* bases (Lyndon, Sirsov) have been proposed in the sequel, Viennot [23] showed that they are all special cases of generalized Hall bases. Underlying their construction is a *unique factorization principle*, which in turn is closely related to Poincare-Birckhoff-Witt bases (of the universal enveloping algebra of a Lie algebra) and Lazard elimination. Formulas for the dual PBW bases ξ_B have been given by Schützenberger, Sussmann[22], Grossman, and Melancon and Reutenauer [15]. For an introductory survey see [11], while [15] elucidates the underlying Hopf algebra structure, and [18] is the principal technical reference for combinatorics of free Lie algebras.

1.3 Available related results

The direct expansion of the logarithm into a formal power series may be simplified using symmetrization [18, 22], but this still does not yield well-defined *"coordinates"* with respect to a basis.

Explicit, but quite *unattractive* formulas for the first 14 coefficients ζ_H in the case of $m = 2$ and a Hall basis are calculated in [10]. This calculation can be automated in a computer algebra system for terms of considerably higher order, but no apparent algebraic structure is discernible. These results suffice for some numerical purposes, but they don't provide much structural insight.

Several new algebraic structures introduced in [19] lead to systematic formulas for ζ_B using spanning sets $\mathcal{B}$ that are smaller than those in [22], but are not bases. These formulas can be refined to apply to Hall-bases, but at the cost of loosing their simple structure. Further recent insights into the underlying algebraic structures may be found in [4, 13].

The introductory survey [11] lays out in elementary terms the close connections between Lazard elimination, Hall-sets, chronological products, and the particularly attractive formula (1.3). These intimate connections suggest that to obtain similarly attractive expressions for ζ_B one may have to start from the very beginning by building bases for free Lie algebras that do not rely on recursive use of Lazard elimination. While it is desirable that any such new bases still restrict to bases of the homogeneous subspaces of the free Lie algebra, we suggest consider balancing the simplicity of the basis for the Lie algebra and structural simplicity of the formulas for the dual objects ζ_B. In particular, consider bases whose elements are not necessarily Lie monomials, but possibly nontrivial linear combinations of iterated Lie brackets of the generators.

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