Chapter 1

Nilpotent bases of distributions

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1.1 Description of the problem

When modelling controlled dynamical systems one commonly chooses individual control variables $u_1, \dots u_m$ which appear *natural* from a physical, or practical point of view. In the case of nonlinear models evolving on $\mathbf{R}^n$ (or more generally, an analytic manifold M^n) that are affine in the control, such a choice corresponds to selecting vector fields $f_0, f_1, \dots f_m \colon M \mapsto TM$, and the system takes the form

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} u_k \, f_k(x). \tag{1.1}$$

From a geometric point of view such a choice appears arbitrary, and the natural objects are not the vector fields themselves, but their linear span. Formally, for a set $\mathcal{F} = \{v_1, \dots v_m\}$ of vector fields define the *distribution spanned by* $\mathcal{F}$ as

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$\Delta_{\mathcal{F}}: p \mapsto \{c_1 v_1(p) + \ldots + c_m v_m(p): c_1, \ldots c_m \in \mathbf{R}\} \subseteq T_p M$. For *systems with drift* f_0, the geometric object is the map $\tilde{\Delta}_{\mathcal{F}}(x) = \{f_0(x) + c_1 f_1(x) + \ldots + c_m f_m(x): c_1, \ldots c_m \in \mathbf{R}\}$ whose image at every point x is an affine subspace of $T_x M$. The geometric character of the distribution is captured by its invariance under the *group of feedback transformations.* In traditional notation (here formulated for systems with drift) these are (analytic) maps (defined on suitable subsets) $\alpha: M^n \times \mathbf{R}^m \mapsto \mathbf{R}^m$ such that for each fixed $x \in M^n$ the map $v \mapsto \alpha(x, v)$ is affine and invertible. Customarily one identifies $\alpha(x, \cdot)$ with a matrix and writes

$$u_k(x) = \alpha_{0k}(x) + v_1 \alpha_{1k}(x) + \ldots v_m \alpha_{mk}(x) \quad \text{for } k = 1, \ldots m. \tag{1.2}$$

This transformation of the controls induces a corresponding transformation of the vector fields defined by $\dot{x} = f_0(x) + \sum_{k=1}^m u_k f_k(x) \stackrel{!}{=} g_0(x) + \sum_{k=1}^m v_k g_k(x)$

$$\begin{array}{lcll} g_0(x) & = & f_0(x)+ & \alpha_{01}(x) f_1(x) + \ldots \alpha_{0m}(x) f_m(x) \\ g_k(x) & = & & \alpha_{k1}(x) f_1(x) + \ldots \alpha_{km}(x) f_m(x), \quad k = 1, \ldots m \end{array} \tag{1.3}$$

Assuming linear independence of the vector fields such feedback transformations amount to *changes of basis* of the associated distributions. One naturally studies the *orbits* of any given system under this group action, i.e. the collection of equivalent systems. Of particular interest are *normal forms*, i.e natural distinguished representatives for each orbit. Geometrically (i.e., without choosing local coordinates for the state x) these are characterized by properties of the Lie algebra $L(g_0, g_1, \ldots g_m)$ generated by the vector fields g_k (acknowledging the special role of g_0 if present).

Recall that a Lie algebra L is called nilpotent (solvable) if its central descending series $L^{(k)}$ (derived series $L^{<k>}$) is finite, i.e. there exists $r < \infty$ such that $L^{(r)} = \{0\}$ ($L^{<r>} = \{0\}$). Here $L = L^{(1)} = L^{<1>}$ and inductively $L^{(k+1)} = [L^{(k)}, L^{(1)}]$ and $L^{<k+1>} = [L^{<k>}, L^{<k>}]$.

The main questions of practical importance are:

Problem 1.
Find necessary and sufficient conditions for a distribution $\Delta_{\mathcal{F}}$ spanned by a set of analytic vector fields $\mathcal{F} = \{f_1, \ldots f_m\}$ to admit a *basis* of analytic vector fields $\mathcal{G} = \{g_1, \ldots g_m\}$ which generate a Lie algebra $L(g_1, \ldots g_m)$ that has a *desirable structure*, i.e. that is **a**. nilpotent, **b**. solvable, or **c**. finite dimensional.

Problem 2.
Describe an algorithm that constructs such a basis $\mathcal{G}$ from a given basis $\mathcal{F}$.

1.2 Motivation and history of the problem

There is an abundance of mathematical problems, which are hard as given, yet are almost trivial when written in the *right* coordinates. Classical examples of finding the *right coordinates* (or, rather, the right bases) are transformations that *diagonalize* operators in linear algebra and functional analysis. Similarly, every system of (ordinary) differential equation is equivalent (via a choice of local coordinates) to the system $\dot{x}_1 = 1, \dot{x}_2 = 0, \ldots \dot{x}_n = 0$ (in the neighbourhood of every point that is not an equilibrium). In control, for many purposes the most convenient form is the controller canonical form (e.g. in the case of $m = 1$) $\dot{x}_1 = u$ and $\dot{x}_k = x_{k-1}$ for $1 < k \leq n$. Every controllable linear system can be brought into this form via feedback and a linear coordinate change. For control systems that are not equivalent to linear systems the next best choice is a polynomial cascade system $\dot{x}_1 = u$ and $\dot{x}_k = p_k(x_1, \ldots, x_{k-1})$ for $1 < k \leq n$. (Both linear and nonlinear cases have natural multi-input versions for $m > 1$.) What makes such linear or polynomial cascade form so attractive for both analysis and design is that trajectories $x(t, u)$ may be computed from controls $u(t)$ by *quadratures* only, obviating the need to solve nonlinear ODEs. Typical examples include pole placement and path planning [11, 16, 19]. In particular, if the Lie algebra is nilpotent (or similarly nice), the general solution formula for $x(\cdot, u)$ as an exponential Lie series [20] (which generalizes *matrix exponentials* to nonlinear systems) collapses and becomes innately manageable.

It is well known that a system can be brought into such polynomial cascade form via a coordinate change if and only if the Lie algebra $L(f_1, \ldots f_m)$ is nilpotent [9]. Similar results for solvable Lie algebras are available [1]. This leaves open only the geometric question about when does a distribution admit a nilpotent (or solvable) basis.

1.3 Related results

In [5] it is shown that for every $2 \leq k \leq (n-1)$ there is a k-distribution Δ on $\mathbf{R}^n$ which does not admit a solvable basis in a neighborhood of zero. This shows the problems of nilpotent and solvable bases are not trivial.

Geometric properties, such as small-time local controllability (STLC) are, by their very nature, unaffected by feedback transformations. Thus conditions for STLC provide valuable information whether any two systems can be feedback equivalent. Typical such information, generalizing the controllability indices of linear systems theory, is contained in the *growth vector*, that is the dimensions of the *derived distributions* which are defined inductively by $\Delta^{(1)} = \Delta$ and $\Delta^{(k+1)} = \Delta^{(k)} + \{[v, w] \colon v \in \Delta^{(k)}, w \in \Delta^{(1)}\}$.

Of highest practical interest is the case when the system is (locally) equivalent to a linear system $\dot{x} = Ax + Bu$ (for some choice of local coordinates). Necessary and sufficient conditions for such exact *feedback linearization* together with algorithms for constructing the transformation and coordinates were obtained in the 1980s [6, 7]. The geometric criteria are nicely stated in terms of the involutivity (closedness under Lie bracketing) of the distributions spanned by the sets $\{(\mathrm{ad}^j f_0, f_1) \colon 0 \leq j \leq k\}$ for $0 \leq k \leq m$.

A necessary condition for exact nilpotentization is based on the observation that every nilpotent Lie algebra contains at least one element that commutes with every other element [4].

A well-studied special case is that of nilpotent systems which can be brought into *chained-form*, compare [16]. This is closely related to *differentially flat* systems, compare [2, 8], which have been the focus of much study in the 1990s. The key property is the existence of an *output function* such that all system variables can be expressed in terms of functions of a finite number of derivatives of this output. This work is more naturally performed using a dual description in terms of exterior differential systems and co-distributions $\Delta^{\perp} = \{\omega\colon M \mapsto T^*M : \langle\omega, f\rangle = 0 \text{ for all } f \in \Delta\}$. This description is particularly convenient when working with small co-dimension $n-m$, compare [12] for a recent survey. (Special care needs to be taken at singular points where the dimensions of $\Delta^{(k)}$ are nonconstant.) This language allows one to directly employ the machinery of *Cartan's method of equivalence* [3] However, a *nice* description of a system in terms of differential forms does not necessarily translate in a straightforward manner into a nice description in terms of vector fields (that e.g. generate a finite dimensional, or nilpotent Lie algebra).

Some of the most notable recent progress has been made in the general framework of Goursat distributions, see e.g. [13, 14, 15, 17, 18, 21] for detailed descriptions, the most recent results and further relevant references.

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