Calculus of nonlinear interconnections with applications

Matthias Kawski
Department of Mathematics
Arizona State University
Tempe, Arizona, 85287
kawski@asu.edu

Abstract

This paper reports progress in the analysis of interconnections of nonlinear systems, employing the chronological formalism. A fundamental observation is the close analogy between feeding outputs of one system back as inputs to another system and the process of Lazard elimination which is at the root of Hall-Viennot bases and chronological products. Possible applications of the algebraic description of interconnections of systems include static and dynamic output feedback, and formal inversions of systems which are of interest for tracking problems. Our description in terms of iterated integral functionals is most readily applicable in the case of nilpotent systems, especially strictly triangular homogeneous systems.

1 Introduction

One of the distinguishing characteristics of *controlled* dynamical systems is their *predestination* to be interconnected: This is particularly prominent when working with input-output descriptions of control systems. But state-space descriptions are just as amenable to analyzing interconnections. Even more generally, behavioral description [31] – which do not *a-priori* assume a distinguished direction of the signal flows – are inherently well-suited for formalizing and analyzing interconnections.

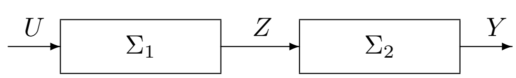
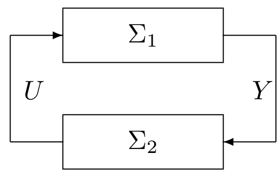


Fig.1 Composition of two systems

Arguably, the most basic interconnection is when the output of one system serves as the input of another system. This point of view is particularly rewarding when used to *decompose* a complex system into an interconnections of smaller, simpler systems. In the easiest case, this decomposition has a cascade from with a distinguished direction of the signal flow.

The Kalman controller normal form [KN] of a linear system may be considered as the ultimate cascade decomposition of a controlled dynamical system. Nonlinear analogues are state-space realizations of solvable or nilpotent systems in polynomial cascade form [6, 15]. Such decompositions into cascades of more simple systems lend themselves particularly well for the systematic design of feedback stabilizers [5, 10, 12, 13, 22].



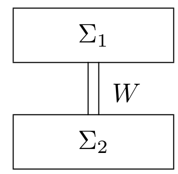


Fig.2 *"Closing the loop"* Fig.3 Interconnection

Static, or even more so, dynamic feedback stabilizers, are an especially important case of interconnections of systems – distinguished by their *"closing the loop"*. The traditional literature distinguishes sharply between the *plant* Σ_1 and the *controller* Σ_2. Behavioral descriptions provide a more symmetric point of view [31]. There are many variations of this theme of interconnections. Here we only mention the practically important problem of tracking, i.e. finding an input u that generates a desired output y. Formally, this is about finding a left inverse Σ_2 of a system Σ_1.

In the context of linear systems a rather complete description of *interconnections* in many different contexts has been established [31]. In addition to many technical difficulties, there are also fundamental structural impediments that stand in the way of developing a similar theory for nonlinear systems. One of the most basic differences is the lack of a natural analogue of the space $L^2([0,\infty))$ that serves as a natural space for both inputs and outputs in the case of linear systems. This is exemplified by the many different notions of stability of nonlinear systems which all collapse to the standard notion in the linear case. For example, in a nonlinear context exponential stability can only be achieved if the system behaves effectively as a nonlinear perturbation of a linear system. In general, one cannot hope for more than rational convergence, like the one exhibited by the solutions $y(t) = y(0)\left(1+(y(0))^2 t\right)^{-1/2}$ of the globally asymptotically stable closed-loop system $\dot{y} = -\frac{1}{2}y^3$. Another fundamental difference is the lack of global solutions, as exhibited in such most benign-looking systems as $\dot{x} = (1+x^2) + 0u$.

This project builds on recent advances on descriptions of a class of nonlinear systems in terms of iterated integral functionals [17, 19]. In this setting, the central object is the Chen-series which dates back to the mid-

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1950s [3]. In the context of controlled dynamical systems it is known as the Chen-Fliess series, based on fundamental work in the 1970s [8]. A series of major advances followed with Sussmann's exponential product expansion of the series [29], and breakthroughs by Melançon and Reutenauer [23, 24] in understanding combinatorial and algebraic features of this series. In the 1990s these results we combined these results with the chronological formalism that Agrachev and Gamkrelidze proposed for an effective calculus of flows of time-varying vector fields [1, 2]. Throughout the last decade a major thrust was to exhibit and understand the connections between algebraic and combinatorial features on one side, and geometric and analytic features on the other side [17, 19]. These include applications to small-time local controllability [21] and optimal control [18], and in particular, to normal forms for nilpotent control systems [19] with immediate uses for homogeneous feedback stabilization.

This paper surveys fundamental questions and provides initial results towards a structural, algebraic analysis of interconnections of nonlinear systems utilizing the chronological framework. The systems under consideration are defined by sets of analytic vector fields on a manifold. As such they can locally be represented in terms of (generally infinite) series of iterated integral operators, or alternatively, as infinite directed products of exponentials of iterated Lie brackets of the system vector fields with iterated integral functionals as coefficients [17]. The converse problem of finding effective algorithmic criteria which I/O systems, or which iterated integral functionals, may be *realized* as vector field systems is subject of continuing research [11, 25]. In our context it is convenient to consider as inputs what usually would be called the integrals (*"primitives"*) of the controls. (The traditional *bounded, measurable controls* are immediately recovered by differentiation.) With this convention, a natural space for both inputs and outputs are the locally absolutely continuous functions (on the real line).

Cascade interconnections are simply compositions, while *"closing the loop"* corresponds to additionally imposing suitable equality constraints. Formally the compositions are obtained by *substituting* Chen-Fliess series – or rather their more sophisticated infinite exponential product expansions – into each other. However, to be of any practical use, it is essential to have effective means to calculate the coefficients in the resulting formal expansions. This is where the chronological formalism is most useful as it takes advantage of specific properties of the Hall-Viennot bases [30], and their intimate relationship with the Lazard elimination process.

A fundamental observation is that the coefficients in the expansion of the composition can be obtained explicitly as polynomial expressions in the coefficients of the original systems. When working with the original Chen Fliess series this comes as no surprise – but the resulting expressions are too unwieldly to be of much use. On the other hand, when working with Hall-Viennot bases one readily recognizes that the formal substitutions yield basis elements only in special cases. In general, one needs to *rewrite* the resulting formal brackets and iterated integral functionals to obtain unique expansions in terms of Hall-Viennot basis elements.

This introduction is followed by a brief survey of the chronological formalism in the setting of nonlinear control systems. The subsequent sections present the main technical problems with initial results and sketches of the proofs (complete technical proofs are part of journal articles under preparation), as well as some illustrations of possible applications.

2 Chronological formalism

The fundamental building block of controlled dynamical systems is the product [1, 2, 17, 19, 29]

$$(U * V)(t) = \int_0^t U(s) \cdot V'(s)\, ds$$

(We assume that all functions are in $\mathcal{U} = \mathcal{AC}_{\rm loc}([0,T)$, i.e. are locally absolutely continuous.) Note that in the case of linear systems this *chronological product* only occurs in the very special form $t*(t*(t*\ldots t*U_j))\ldots)$ which are easily recognized as the *moments* $\int t^k u(t)\,dt$, $k \geq 0$ of the controls $u_k = U_k'$. However, in nonlinear systems one also must account for all possible products of integrals of all orders of the controls. The chronological formalism (together with Hall-Viennot bases) provides an effective mechanism to avoid the redundancies that arise from linear relations between all such possible iterated integral functionals.

Abstractly, a (right) chronological algebra is a linear space $\mathcal{C}$ with a (generally noncommutative, nonassociative) bilinear operation $*\colon \mathcal{C}\times\mathcal{C}\mapsto\mathcal{C}$ that satisfies the (right) chronological identity (for all $r,s,t\in\mathcal{C}$)

$$r*(s*t) = (r*s)*t + (s*r)*t \qquad (1)$$

One immediately verifies that the product (2) equips the space $\mathcal{AC}_{\rm loc}([0,T)$ with a right chronological algebra structure. Using the chronological product the Chen Fliess series [3, 8] may be compactly written as $S = \sum_I U^{*I} X^I$ [19, 28] where $U^{*(Ia)} = U^{*I} * U^a$ and $X^{(Ia)} = X^I X^a$ (for a multi-index, or *word* I, and a *letter* a). It is the unique solution of the *universal control system* (with the initial condition $S(0)=1$)

$$\frac{d}{dt}S = S \cdot \sum_{k=1}^{m} \frac{d}{dt}U^k\; X^k \qquad (2)$$

or more compactly $S = 1 + S * U$ where $U = \sum_{k=1}^m U^k X^k$. Here $\mathcal{X} = \{X^1, X^2, \ldots\}$ is a set of indeterminates and the system *lives* on the free associative (or the free chronological) algebra $\mathcal{A}(\mathcal{X})$ over $\mathcal{X}$. Instead of solving (2) by Picard iteration (which yields the Chen Fliess series) one may alternatively use the variation-of-parameters technique. This yields an infinite exponential product expansion of the solution [29].

The fundamental building block of this construction is the Lazard elimination process: The free Lie algebra $L_{\mathbf{k}}(\mathcal{X})$ (with coefficients in a field $\mathbf{k}$, here $\mathbf{k} = \mathbb{R}$) over a set $\mathcal{X}$ is (for any $a \in \mathcal{X}$) equal to the direct sum of $\mathbf{k} \cdot \{a\}$ and the free Lie algebra generated by the set $\{(\mathrm{ad}^j a, b) : b \in \mathcal{X} \setminus \{a\}, j \geq 0\}$.

The Lazard elimination process not only matches the recursive application of the variation-of-parameters solution technique, but it also yields bases for the free Lie algebra. Specifically, a Hall-Viennot set [30] over a set $\mathcal{X}$ is any strictly ordered subset $\tilde{\mathcal{H}}$ of the set $\mathcal{M}(\mathcal{X})$ of labeled binary trees over $\mathcal{X}$ that satisfies

- $\mathcal{X} \subseteq \tilde{\mathcal{H}}$
- Suppose $a \in \mathcal{X}$ and $w \in \mathcal{H}$ (3)
  Then $(w, a) \in \tilde{\mathcal{H}}$ iff $w < a$ and $a < (w, a)$.
- Suppose $u, v, w, (u, v) \in \tilde{\mathcal{H}}$. Then $(u, (v, w)) \in \tilde{\mathcal{H}}$ iff $v \leq u \leq (v, w)$ and $u < (u, (v, w))$.

The images of Hall-Viennot sets under the canonical map from labeled binary trees to Lie algebras are bases of the free Lie algebra over $\mathcal{X}$. Another distinguishing and very useful property of any Hall-Viennot set $\mathcal{H}$ is that the restriction (to $\mathcal{H}$) of the natural *deparenthesation map* from the labeled binary trees to the free associative algebra over $\mathcal{X}$ is one-to-one. This allows one to use *unparenthesized* Hall words for labeling and in calculations! Sussmann's infinite exponential product expansion of the solution of the universal control system (2) is (for any Hall-Viennot set $\mathcal{H}$ over $\mathcal{X}$)

$$S(T, u) = \overleftarrow{\prod_{H \in \mathcal{H}}} \exp(\beta^H(T, U)[H]) \tag{4}$$

where $[H]$ is the image of the Hall word H under the canonical map into the Lie algebra $L(\mathcal{X})$, and the iterated integral functionals β^H are recursively defined in terms of the chronological product as $\beta_a(T, U) = U_a(T)$ if $a \in \mathcal{X}$. If $H = \underbrace{MM\ldots M}_{m\text{-times}} K \in \mathcal{H}$ and either K is a letter, or the left factor of K is different from M, then

$$\beta_H = \tfrac{1}{m!} \underbrace{(\beta_M * (\beta_M * (\ldots * \beta_M)}_{m\text{-times}} * \beta_K)) \ldots) \tag{5}$$

Almost as an immediate corollary one obtains a normal form (in coordinates) for a free nilpotent system of order r: Index the coordinates by Hall words of length at most r, and the normal form is given by

$$\left\{ \begin{array}{lcll} x_a & = & 1 * U_a & \text{if } a \in \mathcal{X} \\ x_{HK} & = & x_H * x_K & \text{if } H,\ K,\ HK \in \mathcal{H}(\mathcal{X}) \end{array} \right. \tag{6}$$

Note that every nilpotent Lie algebra is the quotient of a free Lie algebra modulo an ideal. Consequently, each nilpotent system can be realized as the image of a normal form such as (6) of a free nilpotent system under a triangular *linear* map – or conversely, each nilpotent system lifts in a natural way to a free nilpotent system of the form (6).

A free nilpotent system has a graded structure which immediately translates into homogeneity properties of the system (6): Any weight function $\delta : \mathcal{X} \mapsto G$ with values in a semi-group G (usually G is taken as the rationals) immediately extends to a homomorphism $\delta : \mathcal{X}^* \mapsto G$ defined for all words I over $\mathcal{X}$. This eventually defines a traditional group of dilations $\Delta_t(x) = (t^{\delta(H_1)} x_{H_1}, t^{\delta(H_2)} x_{H_2}, \ldots t^{\delta(H_s)} x_{H_s})$, parameterized by $t > 0$. (Here $\{H_1, H_2, \ldots H_s\}$ is a basis for a free nilpotent Lie algebra). The system (6) is homogeneous in the usual sense with respect to this group of dilations. Alternatively, using the geometric notion of homogeneity [16] define the generalized *Euler* vector field

$$\nu(x) = \sum_{H \in \mathcal{H}} \delta(H) x_H \frac{\partial}{\partial x_H}. \tag{7}$$

Then the vector fields f_a, $a \in \mathcal{X}$ defined by the rewriting the system (6) in the form $\dot{x} = \sum_{a \in \mathcal{X}} u_a f_a(x)$ satisfy the geometric (coordinate-free) homogeneity criterion $[\nu, f_a] = -\delta(a)\, f_a$.

3 Interconnections

A fundamental question about interconnections of systems asks which properties are preserved: If two systems Σ_1 and Σ_2 have a property P, does their interconnection also have this property P? The linear theory builds heavily on principles such as that cascade interconnections of, say SISO, controllable systems are again controllable SISO systems (under suitable observability / rank conditions). By the same token, stabilizability is preserved. Feedback loops are a different kind of interconnections, and they are typically designed by interconnecting stable systems. For a detailed discussion of general interconnections in the framework of linear behavioral systems see [31].

In the case of nonlinear systems the situation is more delicate. A major source of complications is the lack of a natural analogue of $L^2([0, \infty))$ which serves as a natural space for both inputs and outputs for linear systems. Moreover, many notions of stability (see e.g. [27]) and controllability have evolved which, in general do not agree. Work on relating the properties of interconnections of nonlinear systems to those of the factors goes back a long way, see e.g. [26]. The most promising concept of stability appears to be the one of input-to-state-stability [4]. One of the most simple interconnections simply adds an integrator, or a power-integrator to a nonlinear system. There is a rich literature about this technique in the context of feedback stabilization of nonlinear systems, see e.g. [5, 10, 12, 22, 13].

Consider two affine systems given by collections of analytic vector fields $\mathcal{F} = \{f_0, f_1, \ldots f_{m_1}\}$ and $\mathcal{G} = \{g_0, g_1, \ldots g_{m_2}\}$ on M_1 and M_2 together with analytic output functions $h_1 \in C^\omega(M_1, \mathbb{R}^{m_2})$ and $h_2 \in C^\omega(M_2, \mathbb{R}^p)$. For consideration of a local nature assume w.l.o.g. that $M_1 = \mathbb{R}^{n_1}$ and $M_2 = \mathbb{R}^{n_2}$, and that all systems are initialized at $x(0) = 0$. Often also $n_1 = p$. For our structural analysis it suffices to assume that the controls take values in compact subsets of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. Customarily one writes the systems as

$$\Sigma_1\colon \quad \begin{cases} \dot x &= f_0(x) + \sum_{i=1}^{m_1} u_i f_i(x) \\ y &= h_1(x) \end{cases}$$
$$\Sigma_2\colon \quad \begin{cases} \dot x &= g_0(x) + \sum_{i=1}^{m_2} u_i g_i(x) \\ y &= h_2(x) \end{cases} \tag{8}$$

The cascade interconnection of these systems evolves on the product $M = M_1 \times M_2 = \mathbb{R}^{n_1+n_2}$, again described by an affine system

$$\Sigma = \Sigma_2 \circ \Sigma_1\colon \begin{cases} \dot x &= F_0(x) + \sum_{i=1}^{m_1} u_i F_i(x) \\ y &= H(x) \end{cases} \tag{9}$$

The output $H \in C^\omega(M, \mathbb{R}^p)$ is $H(x_1, \dots x_{n_1+n_2}) = h_2(x_{n_1+1}, \dots x_{n_1+n_2})$. The controlled vector fields F_i, $i = 1, \dots m_1$ are extensions of the fields f_i, $i = 1, \dots m_1$, i.e. if $f_i(x_1, \dots x_{n_1}) = \sum_{j=1}^{n_1} f_i^j(x_1, \dots x_{n_1}) \frac{\partial}{\partial x_j}$, then

$$F_i(x_1, \dots x_{n_1+n_2}) = \sum_{j=1}^{n_1} f_i^j(x_1, \dots x_{n_1}) \frac{\partial}{\partial x_j}. \tag{10}$$

More interesting is the new drift vector field F_0 which assembles the drift and controlled fields of Σ_1 and Σ_2 (writing g_i^j for the components of the vector fields g_i)

$$\begin{aligned} F_0(x_1, \dots x_{n_1+n_2}) &= \sum_{j=1}^{n_1} f_0^j(x_1, \dots x_{n_1}) \frac{\partial}{\partial x_j} + \\ &+ \sum_{j=1}^{n_2} g_0^j(x_{n_1+1}, \dots x_{n_1+n_2}) \frac{\partial}{\partial x_{n_1+j}} \\ &+ \sum_{i=1}^{m_2} \sum_{j=1}^{n_2} h_i(x_1, \dots x_{n_1}) g_i^j(x_{n_1+1}, \dots x_{n_1+n_2}) \frac{\partial}{\partial x_{n_1+j}} \end{aligned} \tag{11}$$

Without any further assumption on the output h_1 one cannot hope that either accessibility or controllability carry over. However, under suitable rank-conditions on the output, i.e. in terms of minimal realizations of compositions of iterated integral functionals, one can prove that the combined system is strongly accessible when the constituent systems are. Regarding accessibility already linear systems show the main problem if no such condition is: Suppose each of $\Sigma_1 = \Sigma_2$ is a chain of two linear integrators $\dot x_1 = u$, $\dot x_2 = x_1$ and the output $h_1(x) = x_1$ is fed as control into the second system. The combined system (9) is not accessible as (using the notation from above) $x_3 - x_2$ is constant along all solutions. In nonlinear systems controllability is a stronger property than accessibility. In this setting define *small-time local controllability* (STLC) for *systems with output*: A system of form (9) is STLC if for all $t > 0$ the reference output $y = 0$ lies in the interior of the set $\{y = H(x(t, U))\colon U \in \mathcal{U}\}$ for all $t > 0$. In the case that H is the identity, this notion agrees with the classical notion of STLC [28].

The composition of a single integrator $\Sigma_1\colon \dot w = v$ (with *output* $u = w$ and control $|v(\cdot)| \le 1$) with the system

$$\Sigma_2 : \begin{cases} \dot x_1 &= u \qquad |u(\cdot)| \le 1 \\ \dot x_2 &= x_1 \qquad y = (x_1, x_2, x_3 - x_4) \\ \dot x_3 &= x_1^4 \\ \dot x_4 &= x_2^2 \end{cases} \tag{12}$$

illustrates the consequences of (via interconnections) imposing constraints on the *variation* of the input: Each system Σ_i with output is STLC. One readily calculates for piecewise constant control $u\colon [0,T] \mapsto [-1,1]]$ taking the values $+1, -1, +1, -1$ on the intervals $[t_{j-1}, t_j]$, respectively, that $x_1(T,u) = x_2(T,u) = x_3(T,u) - x_4(T,u) = y(T,u) = 0$ if the switching times are chosen as approximately $t_0 = 0$, $t_1 = 1.0350$, $t_2 = 3.4848$, $t_3 = 5.8640$, and $t_4 = T = 6.8284$ and that the Jacobian of the map $(t_1, t_2, t_3) \mapsto (x_1, x_2, y)(T, u)$ has full rank at these values for the switching times, and thus it is possible to reach an open neighborhood of $y = (0,0,0)$ by arbitrarily small variations of these switching times. Due to the homogeneity, i.e. since $x_3(\delta T, u_{\varepsilon,\delta}) = \varepsilon^4 \delta^5 x_3(T, u_{1,1}$, $x_4(\delta T, u_{\varepsilon,\delta}) = \varepsilon^2 \delta^5 x_3(T, u_{1,1}$, it follows that $y_3(\delta T, u_{1,\delta}) = \delta^5 y_3(T, u_{1,1}$ and the controllability scales to arbitrarily small times. On the other hand, the composed system with output $\Sigma_2 \circ \Sigma_1$ is not STLC as a consequence of claim 1 in [14]: In particular, if $T < \frac{1}{2}$ then necessarily $|u(T,v)| < \frac{1}{2}$, which implies, see [14], that if $x_1(T,u) = 0$ then $x_3(T,u) \le \frac{3}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot x_4(T,u)$ and consequently $y_3(T,u) = x_3(T,u) - x_4(T,u) \le 0$ for any choice of the control $v\colon [0,T] \mapsto [-1,1]$.

Nonetheless, in the case of single-input-single-output systems, the Hermes condition for small-time local controllability carries over in a similar *provided* the output h_1 satisfies suitable rank conditions (which imply oddness of $\Sigma_2 \circ \Sigma_1$). The key is that cascade interconnections of homogeneous systems are again homogeneous (these serve as nilpotent approximating system utilized in the proofs) – here one naturally also requires that the output h_1 is homogeneous: Suppose that ν_1 and ν_2 are vector fields on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ such that $\dot x = -\nu_i(x)$ are globally asymptotically stable. Using the geometric characterization of homogeneity [16], the systems Σ_i Σ_2 are homogeneous if

$$\begin{array}{rclrcl} [\nu_1, f_i] &=& r_i f_i & \qquad [\nu_2, g_i] &=& s_i g_i \\ \nu_1 h_1^i &=& r_i' h_1^i & \qquad \nu_2 h_2^i &=& s_i' h_2^i \end{array} \tag{13}$$

for some constants r_i, r_i', s_i, s_i'. It is rather straightforward to show that for natural conditions on r_i' and s_i the combined system is again homogeneous, now with respect to the natural vector field $\nu_1 \times \nu_2$ on $\mathbb{R}^{n_1+n_2}$. Consequently, it again has a natural graded structure that is useful for homogeneous feedback stabilization techniques. Compositions of lower triangular systems (i.e. whose Jacobian matrices associated with the system vector fields in specific coordinates) are again lower triangular. Note that such systems may themselves be considered as compositions of sequences of scalar systems. Lower triangular systems arise naturally as nilpotent approximating systems of small-time locally controllable systems [9], and they are amenable to various techniques for constructing explicit feedback stabilizers (see e.g. [5, 12, 22, 13]).

As an immediate corollary, the cascade interconnection of nilpotent systems is again a nilpotent system provided the output h_1 is polynomial in a natural set of *terminal states* of Σ_1. This may be characterized invariantly by demanding that sufficiently high Lie derivatives of h_1 with respect to the fields f_i vanish.

Conversely, every nilpotent system can be realized (in suitable coordinates) as a strictly triangular polynomial system [15]. Recall every free nilpotent systems has (for any fixed choice of a Hall-Viennot basis) the distinguished normal form (6). These give rise to normal forms of general (not necessarily free) nilpotent system via triangular *linear* maps. A main challenge is to analyze how to obtain the normal form of the nilpotent system $\Sigma_2 \circ \Sigma_1$ assuming that both Σ_1 and Σ_2 are given in their respective normal forms. The easiest case is that of free nilpotent SISO-systems when the output h_1 is a projection onto one of the coordinates of maximal degree of homogeneity. In this case the cascade system is automatically in its normal form (note, that in general it will no longer be a free nilpotent system) with respect to some Hall-Viennot basis (which need not agree with the Hall-Viennot bases corresponding to the normal forms of the systems Σ_1 and Σ_2). The key observation is the direct correspondence between considering $h_1(x)$ as a virtual input in the combined system and the Lazard elimination process which underlies the construction of Hall-Viennot bases.

In general, the system as specified by (10) and (3) (in these coordinates) is no longer in its normal form as it will contain terms that are dependent on each other. Consider the example where Σ_1 is chain of three linear integrators and Σ_2 contains a quadratic integrator

$$\Sigma_1\colon \left\{ \begin{array}{rcl} \dot{x}_1 &=& u \\ \dot{x}_2 &=& x_1 \\ \dot{x}_3 &=& x_2 \\ y &=& x_1+x_2+x_3 \end{array} \right. \qquad \Sigma_2\colon \left\{ \begin{array}{rcl} \dot{x}_1 &=& u \\ \dot{x}_2 &=& x_1^2 \\ y &=& x_2 \end{array} \right. \tag{14}$$

The direct expansion of the output H using (3) will contain both the iterated integral functionals

$$\int (\textstyle\int\int u) \cdot (\int\int\int\int u) \quad \text{and} \quad \int (\int\int\int u)^2 \tag{15}$$

No Hall-Viennot basis simultaneously contains words that are associated to these functionals. Consequently, unless one makes additional assumptions on the output h_1, one needs to employ combinatorial rewriting algorithms that reduce each word to a linear combination of Hall words. In this example, the combinatorial manipulations encapsulate the use of the identity

$$\int (\textstyle\int\int u) \cdot (\int\int\int\int u) = \int (\int\int\int u) \cdot \int (\int\int\int\int u) - \int (\int\int\int u)^2 \tag{16}$$

In general, associate to each system its Chen Fliess series expansion – that is the abstract Chen Fliess series [17] evaluated at the data of the systems, i.e. obtained by substituting the vector fields f_i and g_i and the output functions h_i into the formal infinite series. Formally, upon appropriate interpretation of the vector fields according to (10) and (3), the series for $\Sigma_2 \circ \Sigma_1$ must equal the series for Σ_2 into which the series associated to Σ_1 have been substituted for the inputs u_i. Due to the graded structure of the series (*"degrees of homogeneity"*) it suffices to consider nilpotent systems with polynomial output. Moreover, the (preservation of the) graded structure *a-priori* guarantees that the components in the composition are polynomial (with *a-priori* known degrees)in the components of the original system. An alternative point of view focuses directly on the exponential product expansion of the Chen-Fliess series [29]. Here it is critical to pay careful attention to the proper interpretations of the flows $e^{(\int u_\pi) f_\pi}$, $e^{(\int u_\pi) g_\pi}$, and $e^{(\int u_\pi) F_\pi}$ of the iterated Lie brackets of the system vector fields as they *live* on different spaces. Note that the formulas (10) and (3) determine how the flows of F_π on $M_1 \times M_2$ are related to the flows of f_π and g_π on M_1 and M_2, respectively. Formally, the iterated integral coefficients in this expansion are given by the same formula $x_{HK} = x_H * x_K$ as in the normal form of a realization of a free nilpotent system. Consequently, the combinatorial manipulations described above (to bring the composition of free nilpotent systems back into its normal form) provide an explicit formula for the exponential product expression that corresponds to the cascade composition.

Feedback interconnections of two systems Σ_1 and Σ_2 as above provide additional challenges. Among the many possible variations we shall here only briefly consider the case when $p = m_1$ and the output $h_2(x)$ of Σ_2 is fed back as input u for Σ_1. Here we are primarily interested in interpreting and drawing conclusions from the formal series and product expansions as described above. The formal composition of the original Chen Fliess series expansions for each system may be interpreted as a power series expansion for solutions to initial value problems for the *autonomous* closed-loop system. Of primary interest is the expansion resulting from combining the exponential product expansions as these are built on Hall-Viennot bases. For practical applications the most useful form is obtained by partially expanding this infinite exponential product into an infinite series in terms of the Poincaré-Birkhoff-Witt bases corresponding to Hall-Viennot bases. Upon first view, this expansion appears similar to the standard power-series solutions, but it avoids their redundancies by using only the minimal number of independent iterated integrals.

4 Further applications

The previous section of this extended summary already outlined the applicability of this analysis to both theoretical questions such as controllability and practical objectives such as feedback stabilization. Here we shall only briefly mention possible uses related to tracking problems. In its easiest form this asks for a given system Σ_1 to find an input u such that the output of the system y matches a desired reference output. Formally this translates into finding a left inverse Σ_2 for Σ_1, i.e. such that $\Sigma_2 \circ \Sigma_1 = \mathrm{I}$ on the space of all inputs (we usually use $\mathcal{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m)$. Unlike in the case of linear systems, there is no hope that one could invert any highly nonlinear system as clearly the inverse of any power integrator $x(\cdot) \mapsto \int_0^t x^p(s)\,ds$ does not have an analytic inverse for $p > 1$. Thus the only hope is for nonlinear systems whose Jacobian linearization has full rank. However, in this case recent studies [20] suggest that some attractive new formulas can be derived using the formal inverse of the chronological product in the setting of free Lie algebras. For illustration consider the chronological product (also written using the shuffle product which is the symmetrization of the $a \sqcup\!\sqcup b = a * b + b * a$ chronological product)

$$(e + a) * (bc) = ((e + a) \sqcup\!\sqcup b)\, c = bc + abc + bac$$

(using the extension of $*$ to $\mathcal{C}_0 \times \mathcal{C}$) may formally be inverted to yield:

$$\begin{aligned} &(e + a)^{-1} * (bc + abc + bac) \\ &\quad= (e - a + aa - aaa + \ldots) * (bc + abc + bac) \\ &\quad= ((e - a + aa - aaa + \ldots) \sqcup\!\sqcup (b + ab + ba))\, c \\ &\quad= bc. \end{aligned}$$

For a detailed discussion of this formal inversion on the level of universal systems see [20]

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