Combinatorics and algebra of series expansions in nonlinear control

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In the absence of closed-form explicit integrals, series expansions are one of the most useful tools for the analysis of systems that involve noncommuting flows. They are used in essential ways for studying controllability and optimality, and more generally for trajectory analysis and approximation, for designing tracking controls, but they also are of fundamental importance outside control theory, for example in numerical analysis for integration of differential equations with algebraic constraints. We will start with briefly comparing and contrasting a variety of such series expansions, such as the Magnus series, Volterra series, Chen-Fliess series and product expansions thereof, as well as the chronological calculus of Agrachev and Gamkrelidze. While all are closely related, they serve different purposes in different settings.

With the advent of modern high performance computing, it also has become more important to have effective implementations for calculating higher order terms taking advantage of large numbers of redundancies among the terms. This leads to problems of algebraic and combinatorial nature, involving e.g. the Hopf algebras of rooted trees and of labelled binary trees, as well as bases for free Lie algebras, chronological algebras, and Zinbiel algebras. We will discuss the respective advantages and shortcomings of the various formulations, how they map to each other, and their geometric interpretations.

Explicit exponential product expansions of the Chen-Fliess series relying on Hall bases were introduced into control over twenty years ago, and have proven extremely useful since. It is clear from first principles that one alternatively may write such series as an exponential of a single Lie series, which appears very attractive from both a computational point of view as well as for the analysis and design of nonlinear control systems. But in spite of numerous efforts by several authors, variously known under names such as *continuous Campbell Baker Hausdorff formula* or *coordinates of the first kind*, none of the proposed algorithms yield formulas that are anywhere close to the extreme elegance and simplicity of the formulas for the coordinates of the second kind, which using the Zinbiel product may be written as: $\xi_{HK} = \xi_H \star \xi_K$ for Hall words H, K, HK. We have traced this difficulty to the structure of the commonly used Hall bases for free Lie algebras which by construction, based on Lazard elimination, are perfect matches for coordinates of the second kind only. We present some alternative strategies and report on preliminary results which show possible alternatives.

Finally, we will point out how these algebraic and combinatorial objects map to the geometric and analytic properties of control systems. One of the most intriguing observations is that while in control we naturally consider noncommuting flows to lead to Lie algebras, algebraically the formulas appear to live more naturally in a Leibniz algebra – that is, simply speaking, a Lie algebra without the anticommutativity requirement. Geometrically, this is related to connections with not necessarily zero torsion. Further specializations allow various simplifications – special cases include systems that are affine in the control as a subclass of fully nonlinear systems, or second order models as they arise naturally in mechanical systems where certain symmetric products are a popular choice for working with affine connections.

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