MAT 362 Advanced Math for Scientists and Engineers
May 8, 2001 Final Exam name________________

Explain what you are doing. Computer printouts without explanations and formulas scattered over a page without clear logical order will be ignored (zero credit!) It is YOUR responsibility to demonstrate that you have mastered the material of this class. Check ALL results with available computer software!

PDE	VC	all
/100	/100	/200

1.a. Expand the function f given below in a Fourier sine-series $f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{kx}{2}\right)$

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} \le x \le \pi \\ x - \pi & \text{if } \pi \le x \le \frac{3\pi}{2} \\ 2\pi - x & \text{if } \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

b. Sketch the graph of f and overlay the graphs of the approximations f_1, f_3, f_5 and f_7 where

$$f_n(x) = \sum_{k=1}^{n} b_k \sin\left(\frac{kx}{2}\right)$$

c. Write out $f_7(x)$ with 4-digit decimal approximations of the Fourier coefficients b_k.

d. Calculate $\int_0^{2\pi} |f(x)|^2 \, dx$ and $\pi \cdot \sum_{k=1}^{7} b_k^2$ and use these, together with Parseval's identity, to find the relative (*"percentage"*) *mean square error* $\text{rel}_7 = \frac{\int_0^{2\pi} |f(x) - f_7(x)|^2 \, dt}{\int_0^{2\pi} |f(x)|^2 \, dx} \cdot 100\,\%$.

2.a. Use separation of variables and Fourier expansions to solve the (PDE) $u_t = u_{xx}$ in the domain $0 < x < 2\pi$ and $0 < t$, with boundary conditions (BC1,2) $u(0,t) = u(2\pi, t) = 0$ for $0 \le t$, and (BC3) $u(x,0) = f(x)$ for $0 \le x \le 2\pi$, with f as in **1**.

> **Explain how** each of (PDE) and the (BCs) is used in which step. In particular, point out **where** the *eigenvalues* come from, and **why** some constants must be zero, negative, or positive.

b. Write out the first four nonzero terms in the Fourier approximation of the solution $u(x,t)$ with 4-digit decimal approximations of the Fourier coefficients.

c. Sketch the graphs of (approximations) of $u(x,1)$ and $u(\pi,t)$.

d. What physical phenomena/objects may be modeled by this partial differential equation? Explain the practical meaning of each of the boundary conditions?

Bonus: Use calculus to estimate the maximum value of $u(\pi,t)$.

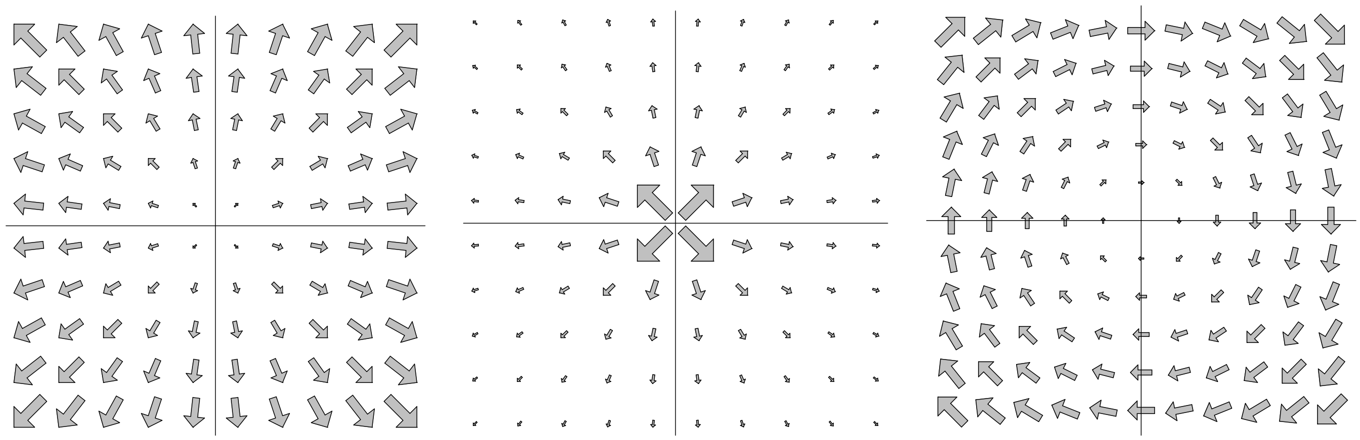
---

turn over, vector calculus on back side ⟶

3.a. Use a parameterization to directly evaluate the line integral $\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{R}$ where C is the upper half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$. **Show details**.

b. Without using parameterizations, evaluate $\int_C 3x^2\vec{\imath} \cdot d\vec{R}$ with C as in part **a**.

c. Let C be the boundary of the triangle with corners $A(3, 0)$, $B(0, 12)$, and $C(-3, 0)$ (oriented counterclockwise). **Use Green's theorem** to rewrite the line integral $\oint_C (3y\vec{\imath} - 2x\vec{\jmath}) \cdot d\vec{R}$ as a double integral and evaluate this integral.



$\vec{F}(x, y)$ $\vec{G}(x, y)$ $\vec{H}(x, y)$

5.a. Which of the three vector fields appear to be linear, and which don't? Explain!

b. Which of the vector fields shown appear to be conservative, and which don't? Explain!

c. Which of the vector fields shown appear to be divergence free, and which don't? Explain!

d. Find a possible formula for each vector field.

e. If possible, find a *potential function* for each vector field. If impossible, **explain why**.

5.a. In which physical setting does the vector field $\vec{F}(x, y, z) = \dfrac{-x\vec{\imath} - y\vec{\jmath} - z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$ arise?

b. Calculate $\dfrac{\partial}{\partial x}\left(\dfrac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right)$ **Show details** of your calculation.

c. Use your result from **b** to show that the divergence of $\vec{F}$ is zero everywhere where $\vec{F}$ is defined.

d. Calculate the flux integral $\oiint_{S_1} \vec{F} \cdot \vec{N}\, dA$ where S_1 is the sphere $x^2 + y^2 + z^2 = a^2$ with $a > 0$.

e. Use the divergence theorem to evaluate $\oiint_{S_2} \vec{F} \cdot \vec{N}\, dA$, where S_2 is the *octahedron* with corners $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, $(0, 0, \pm 2)$, Explain your reasoning in detail.

Bonus. Calculate the flux $\oiint_{S_3} \vec{F} \cdot \vec{N}\, dA$, where S_3 is the triangle with corners $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$. Explain your reasoning in detail.