MAT 362 Advanced Math for Sci & Eng

Orthogonality of the trigonometric functions

The *orthogonality* of the trigonometric functions is the key for making Fourier analysis so immensely effective. This notion of orthogonality, using the *inner product* $< f, g >= \int_0^p f(t)g(t)\,dt$ is a topic of linear algebra (or functional analysis, e.g. infinite dimensional inner product spaces, or Hilbert spaces) – at ASU look for a class in MAT 342 that takes a modern approach!

Nonetheless, the integrals require only elementary calculus, e.g.

- Most texts utilize trigonometric identities such as $\sin\alpha \cdot \cos\beta = \sin\frac{\alpha+\beta}{2} + \sin\frac{\alpha-\beta}{2}$.
- Even easier is to use complex exponentials, e.g. $\sin\alpha \cdot \cos\beta = \frac{e^{i\alpha}-e^{i\alpha}}{2i} \cdot \frac{e^{i\beta}+e^{i\beta}}{2}$ – expand the products and integrate each term separately.
- However, it is integration by parts which most profoundly shows why these functions are orthogonal – it is a direct consequence of them being solutions of the differential equation $(py')' + qy = 0$ with $p \equiv q \equiv 1$. The integrations are very easy, the argument is elegant, and it easily generalizes (using adapted inner products $< f, g >= \ldots$) to other families of special functions (such as Bessel functions) which arise from similar differential equations. The argument is demonstrated below for one of the three integrals.

Integrate by parts twice (assuming $m \geq 0$ and $n > 0$ are integers):

The first time use $\left\{ \begin{array}{rcl} u &=& \cos mt \\ du &=& -m\sin mt\,dt \end{array} \right. \qquad \begin{array}{rcl} dv &=& \sin nt\,dt \\ v &=& -\frac{1}{n}\cos nt\,dt \end{array}$

The second time use $\left\{ \begin{array}{rcl} u &=& \sin mt \\ du &=& m\cos mt\,dt \end{array} \right. \qquad \begin{array}{rcl} dv &=& \cos nt\,dt \\ v &=& \frac{1}{n}\sin nt\,dt \end{array}$

$$\int_{-\pi}^{\pi} \cos mt \cdot \sin nt\,dt = \underbrace{\left(\tfrac{1}{n}\right)\cos mt \cdot (-\cos nt)\Big|_{-\pi}^{\pi}}_{=0 \;\text{(due to periodicity)}} - \left(\tfrac{m}{n}\right)\int_{-\pi}^{\pi} \sin mt \cdot \cos nt\,dt$$

$$= 0 - \underbrace{\left(\tfrac{m}{n^2}\right)\sin mt \cdot \sin nt\Big|_{-\pi}^{\pi}}_{=0 \;\text{(due to periodicity)}} + \left(\tfrac{m^2}{n^2}\right)\int_{-\pi}^{\pi} \cos mt \cdot \sin nt\,dt$$

Thus if $m \neq n$ the integral must be zero. The case of $m = n$ is even easier as the first integration by parts already yields $\int_{-\pi}^{\pi} \cos nt \cdot \sin nt\,dt = -\int_{-\pi}^{\pi} \sin nt \cdot \cos nt\,dt$.

Exercises:

1. Use integration by parts to verify the analogous results for the integrals $\int_{-\pi}^{\pi} \cos mt \cdot \cos nt\,dt$ and $\int_{-\pi}^{\pi} \sin mt \cdot \sin nt\,dt$. Pay special attention to the cases $m = n$, which requires a different argument – clearly as $\cos^2 nt \geq 0$ the integral is also nonzero!
2. Generalize the arguments (in all three cases) to general periods p, e.g. use integration by parts to establish that $\int_0^p \cos\frac{2\pi mt}{p} \cdot \sin\frac{2\pi nt}{p}\,dt$ for all combinations of integers m and n.
