3.6 Green's theorem for linear fields

3.6.1 *Circulation* line integrals of linear fields

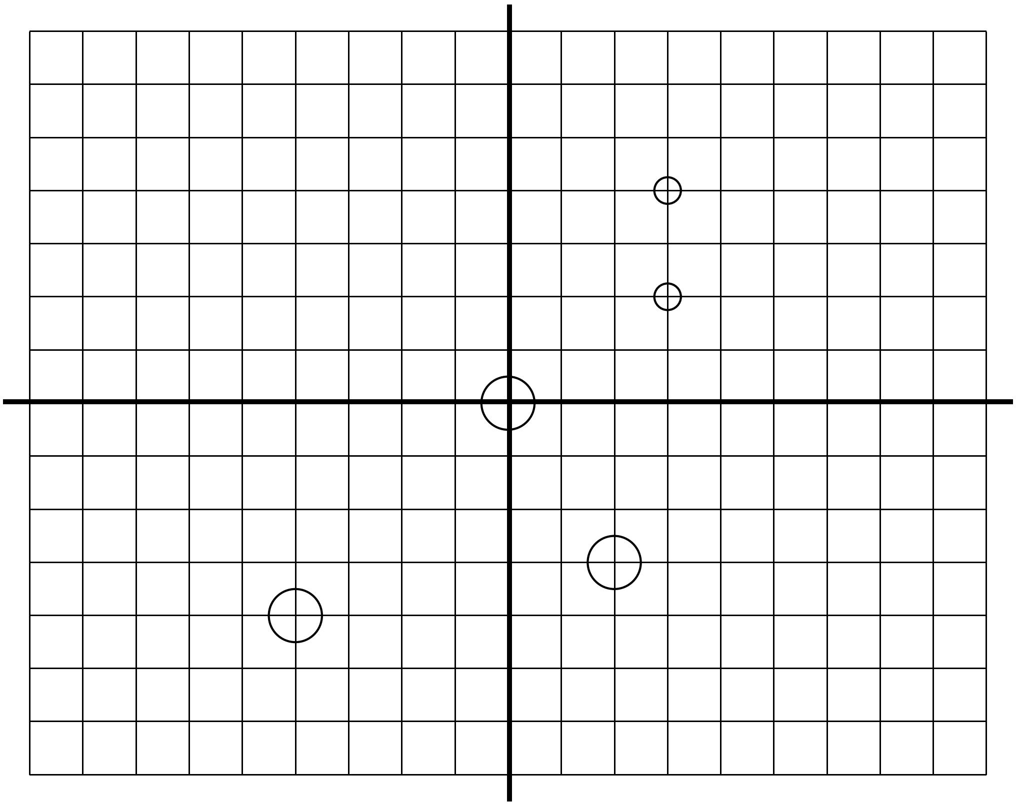
Only very few vector fields are gradient fields. For most pairs of vector fields and closed curves one may expect the line integral to be different from zero. This section begins a closer investigation of such vector fields. This back-door approach will, quite surprisingly lead to the development of a notion of derivative for vector fields, and associated analogues of the fundamental theorem of calculus.
Rather than considering the most general vector fields and general curves, it makes sense to start analyzing the structurally most simple vector fields and curves. Note that every constant vector field is a gradient field. Hence the line integrals of any constant field over any closed curve is zero.

Exercise 3.6.1 *For a constant vector field $\vec{F}(x,y) = a\vec{\imath} + b\vec{\jmath}$ find a potential function $\varphi(x,y)$.*

The structurally next most simple case is that of linear vector fields

$$\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath} \tag{3.1}$$

Exercise 3.6.2 (**Class-exercise**:)
Consider the linear vector field $\vec{L}(x,y) = (3x-2y)\vec{\imath} + (5x+8y)\vec{\jmath}$.
(i) For each of the closed curves depicted below evaluate the line integral $\oint_C \vec{L} \cdot d\vec{r}$.



Add your own triangles, squares, rectangles, semi-and quarter circles, ...

and tabulate the results

Student name	*Contour*	*Line integral*		
	Circle C_1	21.9911		
	Square C_2	28		
	Triangle C_3	3.5		

(ii) Look for patterns. Use the last two columns for additional data as needed. Make conjectures. Test the conjectures by evaluating additional line integrals over other curves, or by modifying the vector field.

Your conjecture should be powerful enough to predict – with only minimal calculation, without paper or calculator! – the value of any line integral of a linear vector field over any closed curve similar to those considered above.

DO NOT READ ON BEFORE COMPLETING THE EXERCISE ABOVE!
Work together. YOU CAN discover some deep math here!
Don't miss the opportunity.

The preceding exercise should have led to some remarkable observations.

Conjecture 3.6.1 *For a linear field the line integral is scaled by the area of the region $\mathcal{R}$ enclosed by the curve C: The ratio of the line integral divided by the area*

$$\frac{\oint_C \vec{L} \cdot d\vec{r}}{(\text{area of } \mathcal{R})} = \text{ constant.} \tag{3.2}$$

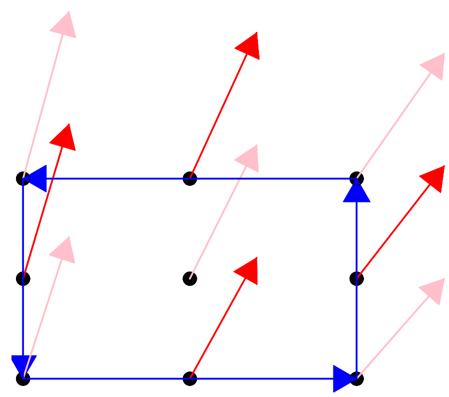
is independent of the location, the shape, and the size of the curve C. The constant equals $(c-b)$ (in above notation) and only depends on the vector field.

Compare this with the slope of a line: The difference quotient (*"rise over run"*)

$$\frac{y_2 - y_1}{x_2 - x_1} = m \text{ is constant.} \tag{3.3}$$

The constant depends only on the choice of the line, but is independent of both the location of the points, and the size of the increment $(x_2 - x_1)$. Of course, this constant is the slope of the line, and it plays a central role in the definition of the derivative of a function of a single variable. In analogy the quotient $(c-b)$ plays a key role in vector calculus, especially in the definition of the curl. It is appropriate to consider $(c-b)$ as a generalization of *"slope"* of a straight line.

The conjecture deserves proof – this special case will lead to Green's theorem! From there we will proceed in small steps, from simple regions to increasingly more general curves. In the next sections we continue with the generalization to nonlinear vector fields. In the end we shall have developed both a compelling new notion of a derivative for vector fields, and a powerful and useful analogue of the fundamental theorem of calculus.



A natural first step is to consider rectangles whose sides are aligned with the coordinate axes.

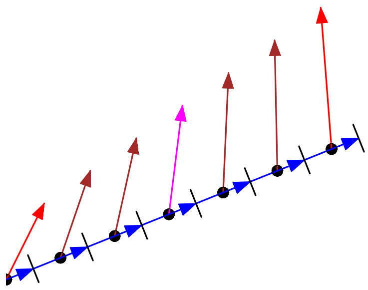
The analysis is much facilitated by the observation that for linear vector fields integrated over line segments the line integral really does not require any calculus at all: Both the midpoint and the trapezoidal rule are exact!

Exercise 3.6.3 *Verify that for a linear vector field $\vec{L}$ and a line segment C from (x_1, y_1) to (x_2, y_2) the line integral equals*

$$\begin{aligned} \oint_C \vec{L} \cdot d\vec{r} &= \vec{L}\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \cdot \Big((x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath}\Big) \\ &= \left(\frac{\vec{L}(x_1, y_1) + \vec{L}(x_2, y_2)}{2}\right) \cdot ((x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath}) \end{aligned}$$

(i) Parameterize the line segment and directly evaluate the line integral for the field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$.

(ii) Argue pictorially, using Riemann sums. Hint: Subdivide the line segment into a number of smaller segments of equal lengths. Group the vectors together analogous to the famous solution supposedly provided by Gauss as a school child when asked to sum $1+2+3+\ldots 99+100 = 100+(1+99)+(2+98)+\ldots(49+51)+50 = 50*100+50 = 5050$.



(iii) Argue geometrically using only the abstract characterization of linearity (i.e without referring to any coordinates) and the vector-form of the parameterization of a line segment $\vec{r}(t) = \vec{r}_1 + t(\vec{r}_2 - \vec{r}_1)$, $t = 0..1$.

Lemma 3.6.2 *The line integral of the linear vector field* $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ *over the edges of the rectangle with corners* $(x_0 \pm \Delta x, y_0 \pm \Delta y)$ *equals*

$$\oint_C \vec{L}\cdot d\vec{r} = (c-b)\cdot 4\Delta x\Delta y \tag{3.4}$$

Proof. (outline). Use the observation about the exactness of the midpoint rule, and calculate:

$$\begin{aligned}
\oint_C \vec{L}\cdot d\vec{r} &= \oint_{C_1} \vec{L}\cdot d\vec{r} + \oint_{C_2} \vec{L}\cdot d\vec{r} + \oint_{C_3} \vec{L}\cdot d\vec{r} + \oint_{C_4} \vec{L}\cdot d\vec{r} \\
&= \vec{L}(x_0, y_0-\Delta y)\cdot(2\Delta x\vec{\imath}) \quad + \quad \vec{L}(x_0+\Delta x, y_0)\cdot(2\Delta y\vec{\jmath}) + \\
&\quad\ \vec{L}(x_0, y_0+\Delta y)\cdot(-2\Delta x\vec{\imath}) \quad + \quad \vec{L}(x_0-\Delta x, y_0)\cdot(-2\Delta y\vec{\jmath}) \\
&= (ax_0 + b(y_0-\Delta y)\vec{\imath} + (\ldots)\vec{\jmath})\cdot(2\Delta x\vec{\imath}) \ + \ ((\ldots)\vec{\imath} + c(x_0+\Delta x) + dy_0\vec{\jmath})\cdot(2\Delta y\vec{\jmath}) + \\
&\quad\ (ax_0 + b(y_0+\Delta y)\vec{\imath} + (\ldots)\vec{\jmath})\cdot(-2\Delta x\vec{\imath}) \ + \ ((\ldots)\vec{\imath} + c(x_0-\Delta x) + dy_0\vec{\jmath})\cdot(-2\Delta y\vec{\jmath}) \\
&= \ldots \\
&= (c-b)\cdot 4\Delta x\Delta y. \quad \blacksquare
\end{aligned}$$

Take a closer look how the scaling by area comes to be: It really consists of two components. On one side, there is the contribution of the increments Δx and Δy to the change in the vector field, e.g. comparing the vector field along the lower edge and along the upper edge. Dividing by the distance between these edges gives a measure of a *rate of change* of the vector field. The other increments Δx and Δy are contributed by the lengths of the curve segments. On a very small scale the vector field is almost constant, and the value of the line integral along each edge is almost directly proportional to the length of the line segment. Taken together, the line integral over the closed contour is scaled by the square of the linear dimension, or by the area of the region enclosed by the curve.

Exercise 3.6.4 *Fill in the omitted details in the calculation above.*

Exercise 3.6.5 *Replace the rectangle above by a triangle with two sides aligned with the coordinate axes. E.g. choose the corners to be* (x_0, y_0), $(x_0+\Delta x, y_0)$, *and* $(x_0, y_0+\Delta y)$. *Carry out the analogous calculation.*

Exercise 3.6.6 *Replace the rectangle above by a triangle with two sides aligned with the coordinate axes. E.g. choose the corners to be* (x_0, y_0), $(x_0 + \Delta x, y_0)$, *and* $(x_0, y_0 + \Delta y)$. *Carry out the analogous calculation.*

Lemma 3.6.3 *Suppose* C *is the curve consisting of the edges of the triangle with corners at* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3) *(oriented counter clockwise). If* $\vec{L}$ *is the linear vector field* $\vec{L}(x, y) = (ax + by)\vec{\imath} + (cx + dy)\vec{\jmath}$, *then*

$$\oint_C \vec{L} \cdot \vec{dr} = (c - b) \cdot (\text{ area of the triangle }) \tag{3.5}$$

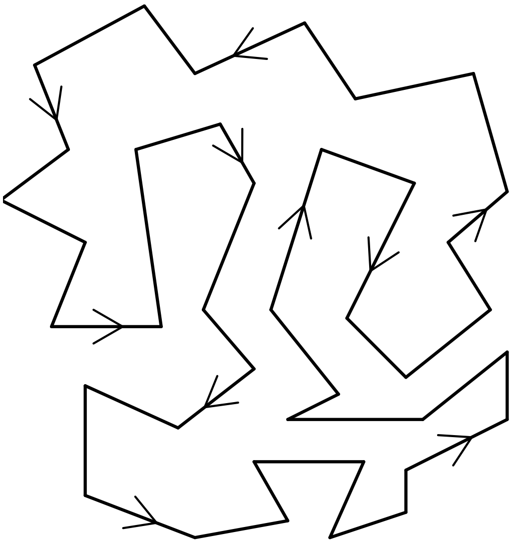
Exercise 3.6.7 *Prove the lemma* (3.6.3) *via direct calculation analogous to that for lemma* (3.6.2)

Exercise 3.6.8 *Using linearity rewrite the vector field* $\vec{L}$ *as* $\vec{L} = \vec{L}_0 + \vec{\Delta L}$ *where* $\vec{L}_0(x, y) = \vec{L}(x_0, y_0)$ *for all* (x, y) *and* $\vec{\Delta L}(x, y) = \vec{L}(x - x_0, y - y_0)$.
(i) Explain why $\oint_C \vec{L}_0 \cdot \vec{dr} = 0$ *and hence* $\oint_C \vec{L} \cdot \vec{dr} = \oint_C \vec{\Delta L} \cdot \vec{dr}$.
(ii) Evaluate $\overrightarrow{\Delta L}$ *at the midpoints of the four edges of the rectangle. Use the midpoint rule to evaluate* $\oint_C \vec{\Delta L} \cdot \vec{dr}$.
(iii) Alternatively, observe that $\overrightarrow{\Delta L}\,(x_0 + \Delta x, y_0 + \Delta y) = \vec{L}(\Delta x, \Delta y)$, *and hence the line integral* $\oint_C \vec{\Delta L} \cdot \vec{dr}$ *equals* $\oint_{C_0} \vec{L} \cdot \vec{dr}$ *where* C_0 *is the curve* C *translated back to the origin (i.e. each point* (x, y) *is moved to the point* $(x - x_0, y - y_0)$*). Thus it suffices to consider rectangles centered at the origin! Repeat the calculations for the case of* $(x_0, y_0) = (0, 0)$.
(iv) Compare these approaches with the calculations above. Which one illustrates best what is going on?
(v) Go even a step further. Using linearity to combine *some of the terms occurring in the calculation. E.g., simplify* $\vec{L}(\Delta x, 0) - \vec{L}(-\Delta x, 0) = 2\Delta x \vec{L}(1, 0)$, *and eventually arrive at*

$$\oint_C \vec{L} \cdot \vec{dr} = \left(\vec{L}(1, 0) \cdot \vec{\jmath} - \vec{L}(0, 1) \cdot \vec{\imath}\right) \cdot (\text{ area of the region } \mathcal{R}). \tag{3.6}$$

Note that this formulation does not make any explicit reference to the coefficients of the vector field when written out in rectangular coordinates. This very geometric point of view is most useful when working with polar or other curvilinear *coordinates.*

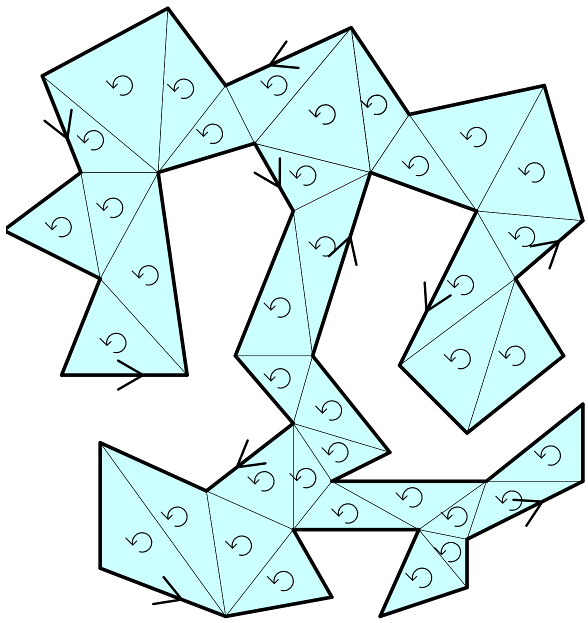
So far we have proven that the conjecture holds true for (linear vector fields integrated over) rectangles that are aligned with the coordinate axes, and, in the exercises, for any triangles. The next step is to establish that the conjecture is true for (linear vector fields integrated over) arbitrary polygonally bounded regions.



It is intuitively clear, but takes a little work to prove rigorously that every region in the plane that is bounded by a polygonal curve, i.e. made up of line segments that do not intersect, can be *triangulated.* This means that the region can be decomposed into a union of a finite number of triangles, any two of which have either no points in common, one corner in common, or one edge in common.

Exercise 3.6.9 *Develop a general argument why any region that is bounded by a polygonal curve can be triangulated as described above. Note, this does not require any calculus. A rigorous typically will require an* induction argument.

Consider a polygonal curve C that encloses a region $\mathcal{R}$ as in the illustration. Suppose that the region is decomposed into a union of triangles (and/or rectangles) labeled $\mathcal{R}_1, \mathcal{R}_2, \ldots \mathcal{R}_n$ (which only meet on their corners or along their edges). Denote the *boundaries* of the regions $\mathcal{R}_k$ by $\mathcal{C}_k = \partial\mathcal{R}_k$, respectively. It is important that all these curves are oriented in a compatible way, i.e. all are oriented counterclockwise.
The area of the entire region $\mathcal{R}$ is just the sum of the areas of the small regions $\mathcal{R}_k$. More exciting is that the sum of the line integrals $\oint_{C_k} \vec{L} \cdot \vec{dr}$ over all the edges of the small regions adds up to just the line integral $\oint_C \vec{L} \cdot \vec{dr}$ over the outside curve: The integrals over all the interior edges cancel as each is traversed once in either direction. Formally this reads:



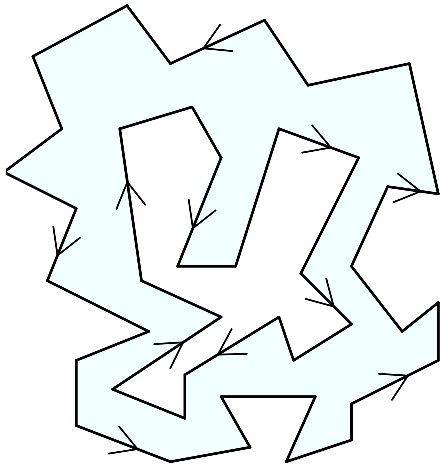
$$
\begin{aligned}
\oint_C \vec{L} \cdot \vec{dr} &= \sum_{k=1}^{n} \oint_{C_k} \vec{L} \cdot \vec{dr} \\
&= \sum_{k=1}^{n} (c-b) \cdot (\text{ area of } \mathcal{R}_k) \\
&= (c-b) \cdot \sum_{k=1}^{n} (\text{ area of } \mathcal{R}_k) \\
&= (c-b) \cdot (\text{ area of } \mathcal{R})
\end{aligned}
$$

This proves the following intermediate result which deserves to be formulated as a lemma.

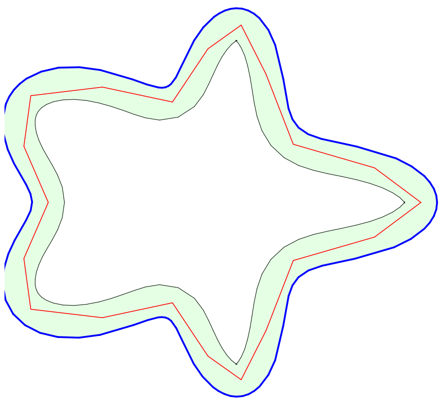
Lemma 3.6.4 *If $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ C is a polygonal curve then*

$$\oint_C \vec{L} \cdot \vec{dr} = (c-b) \cdot (\text{ area of region inside } C\,) \tag{3.7}$$

Exercise 3.6.10 *Consider a region $\mathcal{R}$ that lies between two polygonal curves C_1 and C_2, oriented as shown (i.e. so that the region always lies to the left as one moves along the curve). By carefully going through the steps of the previous arguments, show that it is still true that $\oint_C \vec{L} \cdot \vec{dr} = (c-b)($ area of $\mathcal{R})$ if C now denotes the* generalized curve *consisting of the two* connected *pieces C_1 and C_2.*



Exercise 3.6.11 *Develop a formula for the area of a region that is bounded by a polygonal curve. The formula should take as input the list $(x_1, y_1), \ldots, (x_n, y_n)$ of coordinates of the vertices (corners). Advice: Pick a suitable linear vector field $\vec{L}$ – there are many possibilities – so that $\oint_C \vec{L} \cdot \vec{dr} = 1 \cdot ($ area of $\mathcal{R})$. Utilize the midpoint or trapezoidal rule, each of which is gives the exact value for linear vector fields.*



Finally consider regions that are bounded by piecewise smooth curves C. Any such curve may be arbitrarily closely approximated by a polygonal curve C_P. For the sake of clarity we may assume that the polygonal curve C_P lies entirely *inside* the smooth curve C.
On one side we want the areas of the regions inside the curves C and C_P to be arbitrarily close together.
On the other hand we also want also the line integrals $\oint_C \vec{L} \cdot \vec{dr}$ and $\oint_{C_P} \vec{L} \cdot \vec{dr}$ to be arbitrarily close together. This requires two arguments.

The easy part is that the vector field $\vec{L}$ has almost identical values on *corresponding points* on the respective curves (by hypothesis it is uniformly continuous). It takes a little bit more work to justify that the polygonal curve can also be chosen such that also the *velocity vectors* $\frac{\vec{dr}}{dt}$ are arbitrarily close at *corresponding* points. [[*this deserves an elegant argument – idea is to zoom in very much, so that the smooth curve looks practically straight. The rest is just book-keeping, using uniformity, bounds on $\vec{r}''$* ...]]. This completes the outline of the proof of the conjecture, formulated now as a proposition.

Proposition 3.6.5 *Suppose $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$, and R is a region in the plane (possibly with* "holes"*. Let $\partial R = C$ denote the oriented, piecewise smooth boundary of R. Then*

$$\oint_{\partial R} \vec{L} \cdot \vec{dr} = (c-b) \cdot (\text{ area of } R) \qquad (3.8)$$

In words: For line integrals of linear vector fields over any region that is bounded by piecewise smooth curves: The ratio of the line integral divided by the area of the enclosed region is a constant. This constant is independent of the location, the shape, and the size of the curve. The constant may be considered as an analogue of the slope of a straight line. As such it will be the precursor for a geometric definition of the *scalar curl*, one derivative of vector fields.

This proposition also reaffirms the curl test of the previous section in the special case of linear vector fields.

Corollary 3.6.6 *A linear vector field* $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ *is conservative if and only if* $b = c$.

3.6.2 *Flux* line integrals of linear fields

Practically every step of the previous section has an immediate analogue for flux line integrals in the plane. The following outline, in the form of a list of guided exercises, makes a great *project*. By going once more through the all steps, the overall argumentation, and the role of linearity will become even clearer.

Proposition 3.6.7 *Suppose $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$, and $\mathcal{R}$ is a region in the plane (possibly with* "holes"*. Let $\partial\mathcal{R} = C$ denote the oriented, piecewise smooth boundary of $\mathcal{R}$. Then*

$$\oint_{\partial\mathcal{R}} \vec{L} \cdot d\vec{r} = (c-b) \cdot (\text{ area of } \mathcal{R}\,) \tag{3.9}$$

In words: For line integrals of linear vector fields over any region that is bounded by piecewise smooth curves: The ratio of the line integral divided by the area of the enclosed region is a constant. This constant is independent of the location, the shape, and the size of the curve. The constant may be considered as an analogue of the slope of a straight line. As such it will be the precursor for a geometric definition of the *scalar curl*, one derivative of vector fields.

Exercise 3.6.12 *Repeat the class exercise* (3.6.2) *for flux line integrals $\oint_C \vec{L} \cdot \vec{N}\, ds$. Again start with the specific example $\vec{L}(x,y) = (3x-2y)\vec{\imath} + (5x+8y)\vec{\jmath}$, and evaluate the integral over several curves. Guided by your findings in exercise* (3.6.2)*, experiment with somewhat different vector fields – e.g. experiment who a change of one of the coefficients affects the value of the integral. The final objective should be a carefully stated conjecture.*

Exercise 3.6.13 *In analogy to exercise , verify that the trapezoidal and midpoint rules are exact for flux line integrals of linear vector fields $\vec{L}$ over any line segment C from (x_1,y_1) to (x_2,y_2). I.e. verify (i) by direct calculation using a parameterization, (ii) pictorially as in exercise , or (iii) directly from the definition of linearity.*

$$\begin{aligned} \oint_C \vec{L} \cdot \vec{N}\, ds &= \vec{L}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \cdot \Big(-(y_2-y_1)\vec{\imath} + (x_2-x_1)\vec{\jmath}\Big) \\ &= \left(\frac{\vec{L}(x_1,y_1)+\vec{L}(x_2,y_2)}{2}\right) \cdot (-(y_2-y_1)\vec{\imath} + (x_2-x_1)\vec{\jmath}\vec{\jmath}) \end{aligned}$$

Lemma 3.6.8 *The flux line integral of the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$ over the edges of the rectangle with corners $(x_0 \pm \Delta x, y_0 \pm \Delta y)$ equals*

$$\oint_C \vec{L} \cdot \vec{N}\, ds = (a+d) \cdot \Delta\, \Delta y \tag{3.10}$$

Exercise 3.6.14 *Prove lemma* (3.6.8)*, following the outline of the calculation in the proof of lemma* (3.6.2) *and exercise* (3.6.4)*.*

Exercise 3.6.15 *Carry out an analogous calculation for the case when the rectangle is replaced by a triangle with two sides aligned with the coordinate axes. E.g. choose the corners to be (x_0,y_0), $(x_0+\Delta x, y_0)$, and $(x_0, y_0+\Delta y)$.*

Lemma 3.6.9 *Suppose C is the curve consisting of the edges of the triangle with corners at (x_1,y_1), (x_2,y_2), and (x_3,y_3) (oriented counter clockwise). If $\vec{L}$ is the linear vector field $\vec{L}(x,y) = (ax+by)\vec{\imath} + (cx+dy)\vec{\jmath}$, then*

$$\oint_C \vec{L} \cdot \vec{N}\, ds = (a+d) \cdot (\text{ area of the triangle }) \tag{3.11}$$

Exercise 3.6.16 *Prove the lemma* (3.6.9) *via direct calculation analogous to that for lemma* (3.6.8)

Exercise 3.6.17 *In analogy to exercise* (3.6.8) *use linearity rewrite the vector field* $\vec{L}$ *as* $\vec{L} = \vec{L}_0 + \vec{\Delta L}$ *where* $\vec{L}_0(x,y) = \vec{L}(x_0, y_0)$ *for all* (x,y) *and* $\vec{\Delta L}(x,y) = \vec{L}(x - x_0, y - y_0)$.
(i) Explain why $\oint_C \vec{L}_0 \cdot \vec{N}\, ds = 0$ *and hence* $\oint_C \vec{L} \cdot \vec{N}\, ds = \oint_C \vec{\Delta L} \cdot \vec{N}\, ds$.
(ii) Evaluate $\overrightarrow{\Delta L}$ *at the midpoints of the four edges of the rectangle. Use the midpoint rule to evaluate* $\oint_C \vec{\Delta L} \cdot \vec{N}\, ds$.
(iii) Alternatively, observe that $\overrightarrow{\Delta L}\,(x_0 + \Delta x, y_0 + \Delta y) = \vec{L}(\Delta x, \Delta y)$, *and hence the flux line integral* $\oint_C \vec{\Delta L} \cdot \vec{N}\, ds$ *equals* $\oint_{C_0} \vec{L} \cdot \vec{N}\, ds$ *where* C_0 *is the curve* C *translated back to the origin (i.e. each point* (x,y) *is moved to the point* $(x - x_0, y - y_0)$*). Thus it suffices to consider rectangles centered at the origin! Repeat the calculations for the case of* $(x_0, y_0) = (0,0)$.
(iv) Compare these approaches with the calculations in the two preceding exercises. Which one illustrates best what is going on?
(v) Go even a step further. Using linearity to combine *some of the terms occurring in the calculation. E.g., simplify* $\vec{L}(\Delta x, 0) - \vec{L}(-\Delta x, 0) = 2\Delta x \vec{L}(1,0)$, *and eventually arrive at*

$$\oint_C \vec{L} \cdot \vec{N}\, ds = \left(\vec{\imath} \cdot \vec{L}(1,0) + \vec{\jmath} \cdot \vec{L}(0,1)\right) \cdot (\text{ area of } \mathcal{R}). \tag{3.12}$$

Exercise 3.6.18 *Similar to exercise* (3.6.18) *consider a region* $\mathcal{R}$ *that lies between two polygonal curves* C_1 *and* C_2, *oriented as shown in exercise* (3.6.18). *The region always lies to the left as one moves along the curve, and thus the normal vector always points outward from the region. Note that for the inner curve this means that the outward normal points into the hole. By carefully going through the steps of prior arguments, show that it is still true that* $\oint_C \vec{L} \cdot \vec{dr} = (c - b)(\text{ area of } \mathcal{R})$ *if* C *now denotes the* generalized curve *consisting of the two* connected *pieces* C_1 *and* C_2.

Exercise 3.6.19 *Develop a formula for the area of a region that is bounded by a polygonal curve. The formula should take as input the list* $(x_1, y_1), \ldots, (x_n, y_n)$ *of coordinates of the vertices (corners). Advice: Pick a suitable linear vector field* $\vec{L}$ *– there are many possibilities – so that* $\oint_C \vec{L} \cdot \vec{dr} = 1 \cdot (\text{ area of } \mathcal{R})$. *Utilize the midpoint or trapezoidal rule, each of which is gives the exact value for linear vector fields.*

Proposition 3.6.10 *Suppose* $\vec{L}$ *is the linear vector field* $\vec{L}(x,y) = (ax + by)\vec{\imath} + (cx + dy)\vec{\jmath}$, *and* R *is a region in the plane (possibly with* "holes"*. Let* $\partial R = C$ *denote the oriented, piecewise smooth boundary of* R. *Then*

$$\oint_{\partial R} \vec{L} \cdot \vec{N}\, ds = (a + d) \cdot (\text{ area of } R) \tag{3.13}$$

In words: For flux line integrals of linear vector fields over any region that is bounded by piecewise smooth curves: The ratio of the line integral divided by the area of the enclosed region is a constant. This constant is independent of the location, the shape, and the size of the curve. The constant may be considered as an analogue of the slope of a straight line. As such it will be the precursor for a geometric definition of the *divergence*, another derivative of vector fields.