

Notes on Functional Analysis, MAT 571, Spring 2007

1. TOPOLOGICAL VECTOR SPACES

Definition 1.1. A *topological vector space* (or *TVS*) is a vector space X (over \mathbb{R} or \mathbb{C}) with a Hausdorff topology such that addition and scalar multiplication are continuous.

In other words, $+$: $X \times X \rightarrow X$ and \cdot : $\mathbb{F} \times X \rightarrow X$ are continuous, where the scalar field \mathbb{F} is either \mathbb{R} or \mathbb{C} (in general, if no mention is made of which field is being used, our convention will be to assume that it is \mathbb{C}).

Example 1.2. (1) A normed space is a TVS.
(2) $C(X)$ with the compact-open topology is a TVS.
(3) \mathbb{F}^X with the point-open topology is a TVS.

We note the following:

- For $x_0 \in X$, translation $T_{x_0} : X \rightarrow X$ defined by $T_{x_0}(x) = x + x_0$ is a homeomorphism.
- For $c \in \mathbb{F}$, $c \neq 0$, scaling $M_c : X \rightarrow X$ defined by $M_c(x) = cx$ is a homeomorphism.
- If \mathcal{N} is a neighborhood base at 0, then $x + \mathcal{N}$ is a neighborhood base at x ,

(where for $A, B \subseteq X$ and $C \subseteq \mathbb{F}$ we write $A \pm B = \{x \pm y : x \in A, y \in B\}$, $CA = \{cx : c \in C, x \in A\}$, etc.) In particular, we note that if $U \subseteq X$ is open, then $x + U$ is also open.

We give a simple application of these notions.

Lemma 1.3. *Let X be a TVS, and let U be a neighborhood of 0. Then there is a neighborhood V of 0 such that $V \pm V \subseteq U$.*

Proof. Since $0 + 0 = 0 \in U$, and $+$ is continuous, there are neighborhoods V_1 and V_2 of 0 such that $V_1 + V_2 \subseteq U$. Similarly, since $(-1) \cdot$ is continuous, and $0 + (-1) \cdot 0 \in U$, there are neighborhoods V_3 and V_4 of 0 such that $V_3 - V_4 \subseteq U$. Let $V = V_1 \cap V_2 \cap V_3 \cap V_4$. \square

Definition 1.4. A set $E \subseteq X$ is called *balanced* if $cE \subseteq E$ for all $c \in \mathbb{F}$ with $|c| \leq 1$.

Lemma 1.5. *The balanced open sets form a neighborhood base at 0.*

Proof. Let U be a neighborhood of 0. Since $0_{\mathbb{F}} \cdot 0_X \in U$, and scalar multiplication is continuous, there are open neighborhoods A of $0_{\mathbb{F}}$ and B of 0_X such that $AB \subseteq U$. Let $B_r(0) \subseteq A$, and set $V = B_r(0) \cdot B$. Then $V \subseteq U$. V is open, since $V = \cup_{z \in B_r(0) \setminus \{0\}} z \cdot B$ is a union of open sets. V is balanced, since $|c| \leq 1$ implies that $c \cdot B_r(0) \subseteq B_r(0)$. \square

Proposition 1.6. *Let X and Y be TVS's. Let $T : X \rightarrow Y$ be linear. Then T is continuous if and only if T is continuous at 0.*

Proof. If T is continuous, then it is continuous at 0. Conversely, if T is continuous at 0, and x_a is a net in X converging to x , then $x_a - x \rightarrow 0$ by continuity of the vector space operations. Then by hypothesis we have $T(x_a - x) \rightarrow 0$, and this implies that $Tx_a \rightarrow Tx$. \square

Definition 1.7. Let X and Y be TVS's. We let $L(X, Y)$ denote the space of all continuous linear maps from X to Y ; it is a vector space. $L(X, \mathbb{F})$ is denoted X^* , and is called the *dual space* of X .

We note that X^* might equal $\{0\}$.

Proposition 1.8. *Let X be a TVS, and let $f : X \rightarrow \mathbb{F}$ be a non-zero linear functional. The following are equivalent:*

- (1) f is continuous.
- (2) $\ker(f)$ is closed.
- (3) $\ker(f)$ is not dense in X .

Proof. Homework. □

Definition 1.9. Let X be a TVS. A *linear manifold* in X is a vector subspace of X . A *subspace* is a closed linear manifold.

Remark 1.10. Let X be a vector space and $Y \subseteq X$ a subspace (no topology). The *quotient* $X/Y = \{x + Y : x \in X\}$ is a vector space via the operations specified just before Lemma 1.3 (The reader should verify that these operations are well-defined.) $\pi : X \rightarrow X/Y$ defined by $\pi(x) = x + Y$ is called the *quotient map*. π is a surjective linear map.

Definition 1.11. Let X be a topological space, let Z be a set, and let $\pi : X \rightarrow Z$ be a surjective function. The collection $\{E \subseteq Z : \pi^{-1}(E) \text{ is open in } X\}$ is a topology on Z , called the *quotient topology*. It is the strongest topology making π continuous.

(One can think of Z as being the set of *fibers* $\{\pi^{-1}(z) : z \in Z\}$ of the map π . A subset of X is called *saturated* if it is a union of fibers, i.e. of the form $\pi^{-1}(E)$. Then the quotient topology can be thought of as the set of all open saturated subsets of X .)

Proposition 1.12. Let X be a TVS, let $Y \subseteq X$ be a linear manifold, and let $\pi : X \rightarrow X/Y$ be the quotient map. Let X/Y have the quotient topology. Then addition and scalar multiplication are continuous in X/Y . Moreover, X/Y is a TVS if and only if Y is a subspace.

Proof. Homework. □

Definition 1.13. Let X and Y be TVS's. X and Y are called *isomorphic* if there is $T \in L(X, Y)$ such that T is bijective and T^{-1} is continuous.

Proposition 1.14. Let X be a finite dimensional TVS, with dimension n . Then X is isomorphic to \mathbb{F}^n .

Proof. Let e_1, \dots, e_n be the standard basis of \mathbb{F}^n . Let x_1, \dots, x_n be a basis of X . Define a linear map $T : \mathbb{F}^n \rightarrow X$ by $Te_i = x_i$. Then T is bijective. We first show that T is continuous by showing that it is continuous at 0. Let $V \subseteq X$ be a neighborhood of 0. Then $0 \cdot x_1 + \dots + 0 \cdot x_n \in V$. By continuity of the vector space operations in X there is $D = B_r(0) \subseteq \mathbb{F}$ such that $D \cdot x_1 + \dots + D \cdot x_n \subseteq V$. Let $U = D \times \dots \times D \subseteq \mathbb{F}^n$, a neighborhood of 0 in \mathbb{F}^n . Then $T(U) \subseteq V$, proving that T is continuous at 0.

Now we show that T^{-1} is continuous at 0. First note that it suffices to find a bounded neighborhood U of 0 in \mathbb{F}^n such that $T(U)$ is a neighborhood of 0 in X . This is because then, if W is any neighborhood of 0 in \mathbb{F}^n , there is $c > 0$ such that $cU \subseteq W$. Hence $(T^{-1})^{-1}(W) = T(W) \supseteq T(cU) = cT(U)$ is a neighborhood of 0 in X .

Let U be the open unit ball in \mathbb{F}^n . Then ∂U is compact, so $T(\partial U)$ is compact since T is continuous, hence $T(\partial U)$ is closed. Since T is one-to-one and $0 \notin \partial U$, then $0 \notin T(\partial U)$. Therefore $X \setminus T(\partial U)$ is a neighborhood of 0. Let $V \subseteq X \setminus T(\partial U)$ be a balanced neighborhood of 0 (by Lemma 1.5). We claim that $V \subseteq T(U)$, which will finish the proof. To see this, let $x \in V$. Since T is onto there is $z \in \mathbb{F}^n \setminus \partial U$ such that $Tz = x$. Let $G = \overline{B}_1(0) \subseteq \mathbb{F}$. Then $GV \subseteq V$ since V is balanced. Then $T(Gz) = G \cdot Tz = Gx \subseteq V$, so $Gz \cap \partial U = \emptyset$. Since $0 \in Gz$ and Gz is connected, $Gz \subseteq U$. Therefore $z \in U$, so that $x \in T(U)$. □

Corollary 1.15. *Let X be a TVS, and let $Y \subseteq X$ be a finite dimensional linear manifold. Then Y is a subspace.*

Proof. Homework. □

Remark 1.16. The following remarks will be helpful in giving the proof of the preceding Corollary. A net $(x_a)_{a \in A}$ in a TVS is called a *Cauchy net* if for each neighborhood U of 0 there is $a_0 \in A$ such that for all $b, c \succeq a_0$ we have $x_b - x_c \in U$. Convergent nets are Cauchy. For, let $x_a \rightarrow x$, and let U be a neighborhood of 0. Then there is a neighborhood V of 0 such that $V - V \subseteq U$ (by Lemma 1.3). Choose a_0 such that if $b \succeq a_0$ then $x_b - x \in V$. Then if $b, c \succeq a_0$ we have $x_b - x_c = (x_b - x) - (x_c - x) \in V - V \subseteq U$. It is an easy exercise to show that in \mathbb{F}^n , every Cauchy net is convergent.

2. NORMED SPACES AND THEIR DUALS

This material is covered well by Folland in chapter 5, sections 1 – 3. I'll just include here a few ideas that are only presented by Folland in exercises, and a couple of examples.

Definition 2.1. Let X and Y be normed spaces, and let $T \in B(X, Y)$. The *adjoint* of T is the operator $T^* : Y^* \rightarrow X^*$ defined by: $T^*(g) = g \circ T$.

Lemma 2.2. $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

Proof. First, $\|T^*(g)\| = \|g \circ T\| \leq \|g\| \|T\|$, so that $\|T^*\| \leq \|T\|$. For the other inequality, we use the fact that $T^{**} \circ \kappa_X = \kappa_Y \circ T$, where $\kappa_X : X \rightarrow X^{**}$ is the canonical map. We have

$$\begin{aligned} \|T\| &= \|\kappa_Y \circ T\|, \text{ since } \kappa_Y \text{ is an isometry,} \\ &= \|T^{**} \circ \kappa_X\| \\ &\leq \|T^{**}\| \\ &\leq \|T^*\|, \text{ by the first inequality.} \end{aligned} \quad \square$$

Definition 2.3. Let X be a normed space, and let $Y \subseteq X$ and $Z \subseteq X^*$ be linear manifolds. The *annihilator* of Y is the subspace Y^\perp of X^* defined by $Y^\perp = \{f \in X^* : f|_Y = 0\}$. The *preannihilator* of Z is the subspace ${}^\perp Z$ of X defined by ${}^\perp Z = \{x \in X : \hat{x}|_Z = 0\}$.

We note that Y^\perp and ${}^\perp Z$ are indeed subspaces of X^* and X , and not merely linear manifolds.

Proposition 2.4. *Let X be a normed space.*

- (1) *Let Y be a linear manifold in X . Then ${}^\perp(Y^\perp) = \overline{Y}$.*
- (2) *Let Z be a linear manifold in X^* . Then $({}^\perp Z)^\perp \supseteq \overline{Z}$. Equality holds if Z is reflexive.*

Proof. Since $Y \subseteq {}^\perp(Y^\perp)$, and preannihilators are closed, then $\overline{Y} \subseteq {}^\perp(Y^\perp)$. For the reverse containment, let $x \in X$. If $x \notin \overline{Y}$ then the Hahn-Banach theorem implies that there is $f \in Y^\perp$ such that $f(x) \neq 0$, and hence that $x \notin {}^\perp(Y^\perp)$.

The containment in the second part is easily checked. If X is reflexive, then equality follows from the first part, via the canonical map. □

Theorem 2.5. *Let X be a normed space, and let $Y \subseteq X$ be a linear manifold. Then Y^* and X^*/Y^\perp are naturally isometrically isomorphic.*

Proof. Let $r : X^* \rightarrow Y^*$ be the restriction map. Then $\ker r = Y^\perp$, so there is a unique linear map $r_0 : X^*/Y^\perp \rightarrow Y^*$ such that $r = r_0 \circ \pi$ (where $\pi : X^* \rightarrow X^*/Y^\perp$ is the quotient map). Then r_0 is injective and bounded. The Hahn-Banach theorem implies that r , and hence r_0 , is onto. We will show that r_0^{-1} is isometric. Let $g \in Y^*$. From the Hahn-Banach theorem we know that there is $f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$. If $h \in Y^\perp$, then

$$\|f + h\| \geq \|(f + h)|_Y\| = \|f|_Y\| = \|g\| = \|f\|.$$

$$\text{Hence } \|f + Y^\perp\| \geq \|f\| \geq \|f + Y^\perp\|.$$

$$\text{Hence } \|g\| = \|f\| = \|f + Y^\perp\| = \|r_0^{-1}(g)\|.$$

Therefore r_0 is isometric. □

Theorem 2.6. *Let X be a normed space, and let $Y \subseteq X$ be a subspace. Then $(X/Y)^*$ and Y^\perp are naturally isometrically isomorphic.*

Proof. Homework. (The isomorphism is given by the adjoint of the quotient map $X \rightarrow X/Y$.) □

We now move to some applications of the open mapping theorem and principle of uniform boundedness to Fourier series. We let \mathbb{T} denote the unit circle in \mathbb{C} (the one-dimensional torus), with normalized Lebesgue measure. The coordinate function z generates $C(\mathbb{T})$ as a self-adjoint algebra over \mathbb{C} . The span, \mathcal{A} , of the (integer) powers of z is a self-adjoint subalgebra called the algebra of *trigonometric polynomials*. It is uniformly dense in $C(\mathbb{T})$ by the Stone-Weierstrass theorem. For $f \in L^1(\mathbb{T})$ the n th *Fourier coefficient* of f is the number

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z)z^{-n} dz.$$

We note that $|\widehat{f}(n)| \leq \|f\|_1$, so that $\widehat{f} \in \ell^\infty(\mathbb{Z})$ with $\|\widehat{f}\|_\infty \leq \|f\|_1$. The *Fourier transform* $\widehat{\cdot} : L^1(\mathbb{T}) \rightarrow \ell^\infty(\mathbb{Z})$ is thus a bounded linear map.

Theorem 2.7. (*Riemann-Lebesgue Lemma*) *The range of the Fourier transform is contained in $c_0(\mathbb{Z})$.*

Proof. If $f = \sum a_n z^n$ is a trigonometric polynomial, then $\widehat{f} = (a_n)_{n \in \mathbb{Z}} \in c_0(\mathbb{Z})$. Since $\|\cdot\|_1 \leq \|\cdot\|_\infty$ on \mathbb{T} ,

$$\overline{\mathcal{A}}^{\|\cdot\|_1} = \overline{C(\mathbb{T})}^{\|\cdot\|_1} = L^1(\mathbb{T}).$$

Thus we have

$$L^1(\mathbb{T})^\widehat{\cdot} = (\overline{\mathcal{A}}^{\|\cdot\|_1})^\widehat{\cdot} = \overline{\mathcal{A}}^{\|\cdot\|_\infty} \subseteq c_0. \quad \square$$

Theorem 2.8. *The Fourier transform is injective.*

Proof. Let $f \in L^1(\mathbb{T})$ with $\widehat{f} = 0$. Then $\int_{\mathbb{T}} f p = 0$ for all $p \in \mathcal{A}$. Consider $h = \overline{\text{sgn}(f)} \in L^\infty(\mathbb{T}) \subseteq L^1(\mathbb{T})$. Since $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$ there is a sequence $g_n \in C(\mathbb{T})$ converging to h in $\|\cdot\|_1$. Replacing g_n by $g_n \cdot (\max\{1, |g_n|\})^{-1}$ if necessary, we may assume that $\|g_n\|_\infty \leq 1$. Now, passing to a subsequence, we may assume that $g_n \rightarrow h$ almost everywhere. Choose $p_n \in \mathcal{A}$ with $\|p_n - g_n\|_\infty < 1/n$. Then $p_n \rightarrow h$ almost everywhere, and $\|p_n\|_\infty < 2$ for all n . By the dominated convergence theorem we get

$$\int |f| = \int f h = \int \lim_n f p_n = \lim_n \int f p_n = 0.$$

Therefore $f = 0$ in $L^1(\mathbb{T})$. \square

Theorem 2.9. *There exists a sequence in $c_0(\mathbb{Z})$ that is not the Fourier transform of any function in $L^1(\mathbb{T})$.*

Proof. If this were false, then the Fourier transform would be a continuous bijection between $L^1(\mathbb{T})$ and $c_0(\mathbb{Z})$. By the inverse mapping theorem, it would be an isomorphism. But then $L^1(\mathbb{T})$ and $c_0(\mathbb{Z})$ would have isomorphic duals. This is impossible, since the dual of $L^1(\mathbb{T})$ is $L^\infty(\mathbb{T})$, which is inseparable, while the dual of $c_0(\mathbb{Z})$ is $\ell^1(\mathbb{Z})$, which is separable. \square

We now apply the principle of uniform boundedness to the problem of convergence of Fourier series. For $f \in L^1(\mathbb{T})$, the n th Fourier polynomial of f is denoted $s_n(f, \cdot) \in C(\mathbb{T})$, and is given by

$$\begin{aligned} s_n(f, z) &= \sum_{j=-n}^n \widehat{f}(j) z^j \\ &= \sum_{j=-n}^n \int_{\mathbb{T}} f(w) w^{-j} dw z^j \\ &= \int_{\mathbb{T}} f(w) \left(\sum_{j=-n}^n (zw^{-1})^j \right) dw. \end{aligned}$$

We define the *Dirichlet kernel* by

$$\begin{aligned} D_n(z) &= \sum_{j=-n}^n z^j = z^{-n} \sum_{j=0}^{2n} z^j = z^{-n} \frac{z^{2n+1} - 1}{z - 1} = \frac{z^{n+1} - z^{-n}}{z - 1} \\ &= \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{e^{i(n+1/2)t} - e^{-i(n+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{\sin(n+1/2)t}{\sin t/2}, \end{aligned}$$

where we have identified (\mathbb{T}, dz) with $([-\pi, \pi], dt/2\pi)$ by replacing z with e^{it} . We will write $s_n(f, t)$ instead of $s_n(f, e^{it})$ in this context, and expect that the reader will be able to figure out what we mean (!!). Thus $s_n(f) = f * D_n$:

$$s_n(f, t) = \int_{\mathbb{T}} f(s) D_n(t - s) ds.$$

For fixed n and t this is a linear functional of f . We will consider it as a functional on $C(\mathbb{T})$. Its norm in $C(\mathbb{T})^*$ is given by the L^1 -norm of the function $s \mapsto D_n(t - s)$, which is just $\|D_n\|_1$. The numbers $\|D_n\|_1$ are called the *Lebesgue constants*.

There is an old question: if $f \in C(\mathbb{T})$, does the Fourier series of f converge to f ? This is an important question no matter what sense of convergence you have in mind. Classically, people thought that, at least, the series should converge pointwise to f . A lot of very hard analysis was brought to bear on this problem. One of the great pleasures of ‘soft’ analysis is to give an easy, negative resolution of the problem without presenting an explicit counterexample. It goes like this: if $\lim_n s_n(f, t) = f(t)$ for all continuous f , then the family $\{s_n(\cdot, t) : n \in \mathbb{N}\}$ is pointwise bounded on $C(\mathbb{T})$. By the principle of uniform boundedness,

it must be uniformly bounded. As we have seen, this is equivalent to the boundedness of the Lebesgue constants. However it is a routine exercise to show that the Lebesgue constants tend to infinity like $\log n$. In fact, we get the same contradiction if $\lim_n s_n(f, t) = f(t)$ for all f in some subset of $C(\mathbb{T})$ of the second category. Therefore for fixed t , the set of continuous functions whose Fourier series converges to the function at t is a set of the first category. Therefore, if z_1, z_2, \dots is any sequence of points in \mathbb{T} , most continuous functions (i.e. the complement of a set of the first category) have Fourier series that diverge at each point of the sequence.

3. CONVEXITY

Definition 3.1. Let X be a vector space, and let $E \subseteq X$. E is *convex* if for all $x, y \in E$, and for all $t \in [0, 1]$, we have $(1 - t)x + ty \in E$. If $x_1, \dots, x_n \in E$, a *convex combination* of x_1, \dots, x_n is a linear combination $t_1x_1 + \dots + t_nx_n$ where $t_i \geq 0$ for all i and $t_1 + \dots + t_n = 1$.

Lemma 3.2. (1) *If E is convex then E is closed under the formation of convex combinations.*

(2) *Any intersection of convex sets is again convex.*

Proof. The first statement is proved by induction. The inductive step is:

$$t_1x_1 + \dots + t_{n+1}x_{n+1} = (1 - t_{n+1}) \left(\frac{t_1}{1 - t_{n+1}}x_1 + \dots + \frac{t_n}{1 - t_{n+1}}x_n \right) + t_{n+1}x_{n+1}.$$

The second statement is immediate from the definition. \square

Definition 3.3. If $E \subseteq X$, the *convex hull* of E is defined by

$$\text{co}(E) = \bigcap \{A : E \subseteq A \text{ and } A \text{ is convex}\}.$$

Lemma 3.4. *$\text{co}(E)$ equals the set of all convex combinations of elements of E .*

Proof. \subseteq : The set of all convex combinations is a convex set (an easy verification). Since it contains E , it contains $\text{co}(E)$.

\supseteq : By Lemma 3.2, $\text{co}(E)$ contains all convex combinations of its elements, and hence all convex combinations of elements of E . \square

Definition 3.5. A *locally convex topological vector space*, or *LCTVS*, is a TVS having a neighborhood base at 0 consisting of convex open sets.

Lemma 3.6. *A LCTVS has a neighborhood base at 0 consisting of balanced convex open sets.*

Proof. First note that if $E \subseteq X$ is open, then also $\text{co}(E)$ is open. For, if $t_i \geq 0$ and $\sum_i t_i = 1$, then (assuming, without loss of generality, that $t_1 \neq 0$)

$$t_1E + \dots + t_nE = \bigcup_{x_2, \dots, x_n \in E} \left(t_1E + \sum_{i=2}^n t_i x_i \right)$$

is a union of open sets.

Now let U be an open convex neighborhood of 0. Let $V \subseteq U$ be a balanced open set. Let $W = \text{co}(V)$. Then $W \subseteq U$ and W is convex and open. We claim that W is balanced. To see this, let $z \in W$ and $c \in \mathbb{F}$ with $|c| \leq 1$. Then $z = t_1x_1 + \dots + t_nx_n$ for some $x_i \in V$ and $t_i \geq 0$, $\sum t_i = 1$. Then $cx_i \in V$ since V is balanced. So $cz = \sum t_i(cx_i) \in \text{co}(V) = W$. \square

Why are we concerned with local convexity? Roughly, the motivation goes like this. When studying a TVS, we wish to have a rich dual space. But not all TVS's have a substantial dual — some have no non-zero continuous linear functionals at all. The question arises, which TVS's have “sufficiently many” continuous linear functionals? The Hahn-Banach theorem presents a connection between continuous linear functionals and continuous seminorms. Namely, if f is a continuous linear functional, then $|f|$ is a continuous seminorm. Conversely, if p is a continuous seminorm, then the Hahn-Banach theorem provides the existence of continuous linear functionals (dominated by p). The next insight is that given a continuous seminorm p , the set $\{p < 1\}$ is balanced, convex and open. Thus if a TVS has ‘many’ continuous linear functionals, then it has ‘many’ balanced convex open sets. Our next goal is the converse of this: how do we use a balanced convex open set to produce a continuous seminorm? The key idea is that of the *Minkowski functional*. First we formalize some of the ideas sketched above.

Remark 3.7. Let X be a vector space, and let p be a seminorm on X . Then $\{p < 1\}$ is a balanced convex set.

Lemma 3.8. Let X be a TVS, and let p be a seminorm on X . Then p is continuous if and only if $\{p < 1\}$ is open.

Proof. The forward direction is immediate. For the reverse implication, suppose that $\{p < 1\}$ is open. For any $a > 0$, $\{p < a\} = a \cdot \{p < 1\}$ is open. We claim that $\{p > a\}$ is also open. For this, let $p(x) > a$, and put $\varepsilon = p(x) - a$. If $y \in X$ is such that $p(y) < \varepsilon$ then $p(x + y) \geq p(x) - p(y) > p(x) - \varepsilon > a$. Therefore $x + \{p < \varepsilon\} \subseteq \{p > a\}$, proving that $\{p > a\}$ is open. Therefore $p^{-1}((a, b))$ is open for all $a < b$, so p is continuous. \square

Definition 3.9. Let X be a vector space, and let $U \subseteq X$. U is *absorbing* if $X = \bigcup_{t>0} tU$. If U is an absorbing set, the *gauge* (or *Minkowski functional*) of U is $p_U : X \rightarrow [0, \infty)$ defined by

$$p_U(x) = \inf\{t > 0 : x \in tU\}.$$

Example 3.10. (1) $p_X = 0$.

(2) If X is a normed space, and U is the open unit ball, then $p_U = \|\cdot\|$.

Theorem 3.11. Let X be a vector space, and let U be balanced, convex, and absorbing. Then p_U is a seminorm, and

$$\{p_U < 1\} \subseteq U \subseteq \{p_U \leq 1\}.$$

Proof. First note that from the definition, an absorbing set must contain 0. It then follows from the definition of p_U that $p_U(0) = 0$. If $x \in X$ and $s > 0$, then

$$\begin{aligned} p_U(sx) &= \inf\{t > 0 : sx \in tU\} \\ &= \inf\{t > 0 : x \in \frac{t}{s}U\} \\ &= \inf\{rs > 0 : x \in rU\} \\ &= s \cdot p_U(x). \end{aligned}$$

If $\lambda \in \mathbb{F} \setminus \{0\}$, then $\lambda x \in tU$ if and only if $|\lambda|x \in tU$, since U is balanced. Therefore $p_U(\lambda x) = p_U(|\lambda|x) = |\lambda|p_U(x)$.

We now verify the triangle inequality. Let $x, y \in X$, and let $p_U(x) + p_U(y) < c$. Then $p_U(y) < c - p_U(x)$, so there is $0 < t < c - p_U(x)$ such that $y \in tU$. Then also $p_U(x) < c - t$, so there is $0 < s < c - t$ such that $x \in sU$. Then

$$x + y \in sU + tU = (s + t) \left(\frac{s}{s+t}U + \frac{t}{s+t}U \right) \subseteq (s + t)U,$$

by convexity of U . Therefore $p_U(x + y) \leq s + t < c$. It follows that $p_U(x + y) \leq p_U(x) + p_U(y)$.

Finally, if $x \in \{p_U < 1\}$ then $p_U(x) < 1$, so there is $t < 1$ such that $x \in tU$, hence $x \in U$ since U is balanced. If $x \in U$ then $1 \cdot x \in U$, so $\inf\{t > 0 : x \in tU\} \leq 1$. Therefore $p_U(x) \leq 1$. \square

Theorem 3.12. *Let X be a TVS. The map $U \mapsto p_U$ is a bijection between the set of balanced convex open subsets of X , and the set of continuous seminorms on X .*

Proof. If U is nonempty, balanced, convex and open, then p_U is a seminorm by Theorem 3.11. If $x \in U$ then $1 \cdot x \in U$. Since U is open, there is $t < 1$ such that $x \in tU$. Therefore $p_U(x) < 1$. Again by Theorem 3.11, we have that

$$(1) \quad U = \{p_U < 1\}.$$

By Lemma 3.8, p_U is continuous. Therefore the map in the statement is defined. It follows from (1) that the map in the statement is one-to-one. To see that it is onto, let p be a continuous seminorm on X . Put $U = \{p < 1\}$. Then U is balanced, convex and open. Therefore by (1), $U = \{p_U < 1\}$. Then for $c > 0$ we have

$$\{p < c\} = c\{p < 1\} = c\{p_U < 1\} = \{p_U < c\}.$$

It follows that $p = p_U$. \square

Definition 3.13. Let X be a vector space, and let \mathcal{E} be a family of seminorms on X . \mathcal{E} is called *nondegenerate* if for every non-zero $x \in X$ there exists $p \in \mathcal{E}$ such that $p(x) \neq 0$.

Proposition 3.14. *Let X be a LCTVS. Then the family of all continuous seminorms on X is nondegenerate.*

Proof. Let $x \in X$ with $x \neq 0$. Let U be a balanced convex open set with $x \notin U$. By Theorem 3.12, p_U is a continuous seminorm. Since $x \notin U$, Theorem 3.11 implies that $p_U(x) \geq 1$, so $p_U(x) \neq 0$. Therefore the continuous seminorms form a nondegenerate family. \square

Now we will see how to use seminorms to define locally convex topologies.

Theorem 3.15. *Let X be a vector space, and let \mathcal{E} be a nondegenerate family of seminorms on X . For $p_1, \dots, p_n \in \mathcal{E}$ and $\varepsilon > 0$ let*

$$V(p_1, \dots, p_n; \varepsilon) = \{x \in X : p_i(x) < \varepsilon, 1 \leq i \leq n\}.$$

Let \mathcal{B} be the set of all translates of such sets by elements of X . Then

- (1) \mathcal{B} is a base for a locally convex vector space topology on X .
- (2) p is continuous for all $p \in \mathcal{E}$.
- (3) If q is a seminorm on X , then q is continuous if and only if there are $p_1, \dots, p_n \in \mathcal{E}$, and $C > 0$, such that $q \leq C(p_1 + \dots + p_n)$.

Proof. (1): Let $B_1, B_2 \in \mathcal{B}$, and let $x \in B_1 \cap B_2$. We must find $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Let $B_1 = y + V(p_1, \dots, p_m; \delta)$ and $B_2 = z + V(q_1, \dots, q_n; \varepsilon)$. By our assumption, $p_i(x - y) < \delta$ for $1 \leq i \leq m$ and $q_j(x - z) < \varepsilon$ for $1 \leq j \leq n$. Let $\gamma > 0$ with

$$\gamma < \min_{i,j} \{ \delta - p_i(x - y), \varepsilon - q_j(x - z) \}.$$

Let $B_3 = x + V(p_1, \dots, p_m, q_1, \dots, q_n; \gamma)$. Then $x \in B_3$. For $w \in B_3$ we have

$$p_i(w - y) \leq p_i(w - x) + p_i(x - y) < \gamma + p_i(x - y) < (\delta - p_i(x - y)) + p_i(x - y) = \delta,$$

for all $1 \leq i \leq m$. Therefore $w \in B_1$. Similarly, $w \in B_2$. Hence $B_3 \subseteq B_1 \cap B_2$. Therefore \mathcal{B} is a base for a topology on X .

We show that this topology is Hausdorff. Let $x \neq y$. Choose $p \in \mathcal{E}$ such that $p(x - y) \neq 0$. Let $\varepsilon = \frac{1}{2}p(x - y)$. Then $x + V(p; \varepsilon)$ and $y + V(p; \varepsilon)$ are disjoint neighborhoods of x and y .

Now we show that addition and scalar multiplication are continuous. Let $x, y \in X$, and let U be a neighborhood of $x + y$. Choose $p_1, \dots, p_n \in \mathcal{E}$ and $\varepsilon > 0$ such that $x + y + V(p_1, \dots, p_n; \varepsilon) \subseteq U$. Then

$$V(p_1, \dots, p_n; \varepsilon/2) + V(p_1, \dots, p_n; \varepsilon/2) \subseteq V(p_1, \dots, p_n; \varepsilon),$$

and hence

$$(x + V(p_1, \dots, p_n; \varepsilon/2)) + (y + V(p_1, \dots, p_n; \varepsilon/2)) \subseteq x + y + V(p_1, \dots, p_n; \varepsilon).$$

Therefore addition is continuous. We leave the verification that scalar multiplication is continuous as an exercise.

Since the basic open sets are convex, the topology is locally convex.

(2): For each $p \in \mathcal{E}$, $\{p < 1\} \in \mathcal{B}$ and hence is open. By Lemma 3.8, p is continuous.

(3): Suppose that q is a continuous seminorm on X . Then $\{q < 1\}$ is a neighborhood of 0. Therefore it contains a basic open set $V(p_1, \dots, p_n; \varepsilon)$. Then for $x \in X$, if $p_i(x) < \varepsilon$ for all i then $q(x) < 1$. Now let $t > 0$. If $\sum_{i=1}^n p_i(x) < \varepsilon t$ then $p_i(x/t) < \varepsilon$ for all i , and hence $q(x/t) < 1$, hence $q(x) < t$. Thus we see that $q \leq \varepsilon^{-1} \sum_{i=1}^n p_i$.

Conversely, suppose that q is a seminorm such that $q < C \sum_{i=1}^n p_i$ for some $p_1, \dots, p_n \in \mathcal{E}$ and $C > 0$. We will show that $\{q < 1\}$ is open, proving continuity of q by Lemma 3.8. So let $q(x) < 1$. We want to choose $\delta > 0$ such that $x + V(p_1, \dots, p_n; \delta) \subseteq \{q < 1\}$. If $p_i(x - y) < \delta$ for all i , then

$$q(y) \leq q(x) + q(x - y) \leq q(x) + C \sum_{i=1}^n p_i(x - y) < q(x) + Cn\delta.$$

So it is enough that $q(x) + Cn\delta \leq 1$. We may thus choose $\delta = (1 - q(x))/Cn$. \square

Definition 3.16. The topology provided by Theorem 3.15 is called the topology *induced by* \mathcal{E} .

Example 3.17. Let X be a locally compact Hausdorff space, and consider the vector space $C(X)$ of all continuous complex-valued functions on X . For a compact set $K \subseteq X$, define a seminorm p_K on $C(X)$ by $p_K(f) = \sup_{x \in K} |f(x)|$. The topology induced by the family $\{p_K : K \subseteq X \text{ compact}\}$ is the compact-open topology.

Corollary 3.18. Let X be a vector space, let \mathcal{E} be a nondegenerate family of seminorms on X , and let X have the topology induced by \mathcal{E} . Let $\tilde{\mathcal{E}}$ be the family of all continuous seminorms on X . Then the topology induced by $\tilde{\mathcal{E}}$ equals that induced by \mathcal{E} .

Proof. It is clear that $\tilde{\mathcal{E}}$ induces a finer topology than \mathcal{E} . Conversely, if $q \in \tilde{\mathcal{E}}$, then by Theorem 3.15(3) there are $p_i \in \mathcal{E}$ and $C > 0$ such that $q \leq C \sum_{i=1}^n p_i$. But then $V(p_1, \dots, p_n; \frac{\varepsilon}{nC}) \subseteq V(q; \varepsilon)$. Therefore the base defined by \mathcal{E} in Theorem 3.15(1) is already a base for the topology induced by $\tilde{\mathcal{E}}$. \square

Corollary 3.19. *Let X be a TVS. Then X is locally convex if and only if the topology of X is induced from the family of all continuous seminorms on X .*

Proof. Let \mathcal{E} be the set of all continuous seminorms on X . Suppose that X is locally convex, and let U be a balanced convex open set. Then p_U is continuous, by Lemma 3.8, so that $p_U \in \mathcal{E}$. Then U is a basic open set for the topology induced by \mathcal{E} . On the other hand, if U is a basic open set for the topology induced by \mathcal{E} , then by its definition it is open for the topology of X .

Conversely, if the topology of X is induced from \mathcal{E} , then since it is a Hausdorff topology, \mathcal{E} must be nondegenerate. Therefore the topology is locally convex by Theorem 3.15. \square

4. WEAK TOPOLOGIES

We next turn to *weak topologies*, that is, locally convex topologies defined by spaces of linear functionals.

Definition 4.1. Let X be a vector space, and let Y be a vector subspace of the space of all linear functionals on X . Suppose that Y separates the points of X . Then $\{|f| : f \in Y\}$ is a nondegenerate family of seminorms on X . The topology induced by this family is called the $\sigma(X, Y)$ -topology, or the Y -topology on X . It is a locally convex topology by Theorem 3.15.

Remark 4.2. Here are two characterizations of the $\sigma(X, Y)$ -topology.

(1) The basic (balanced, convex) open neighborhoods of 0 are of the form

$$V(f_1, \dots, f_n) = \{x \in X : |f_i(x)| < 1, 1 \leq i \leq n\},$$

where $f_1, \dots, f_n \in Y$.

(2) A net x_a converges to x in the $\sigma(X, Y)$ -topology if and only if $f(x_a) \rightarrow f(x)$ for all $f \in Y$.

Remark 4.3. Let Y be a space of linear functionals on X . Then X is a space of linear functionals on Y , by evaluation. Moreover, as functionals on Y , X necessarily separates the points of Y . Thus there is an X -topology on Y . Sometimes the symmetric nature of the relationship between X and Y is formalized as the notion of a *dual pair*. A *dual pair* is a pair of vector spaces X and Y , together with a bilinear pairing $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{F}$ such that for each $x \in X$ there is $y \in Y$ such that $\langle x, y \rangle \neq 0$, and for each $y \in Y$ there exists $x \in X$ such that $\langle x, y \rangle \neq 0$.

Example 4.4. Let X be a normed space. The $\sigma(X, X^*)$ -topology is called the *weak topology* on X .

Proposition 4.5. *Let X be vector space, and let Y be a space of linear functionals on X that separates points. Then $(X, \sigma(X, Y))^* = Y$.*

Proof. Every element of Y is $\sigma(X, Y)$ -continuous by definition. Conversely, let $g : X \rightarrow \mathbb{F}$ be $\sigma(X, Y)$ -continuous. By Theorem 3.15(3) there are $f_1, \dots, f_n \in Y$ such that $|g| \leq \sum_{i=1}^n |f_i|$. We note that it follows that $\ker g \supseteq \bigcap_{i=1}^n \ker f_i$. If any f_i is a linear combination of the others,

we may use that combination and the triangle inequality to eliminate it. Thus we may assume that the f_i 's are linearly independent. Define $T : X \rightarrow \mathbb{F}^n$ by $Tx = (f_1(x), \dots, f_n(x))$. Since the f_i 's are independent, we know that T is onto. Choose $x_i \in X$ with $Tx_i = e_i$. Then we have $f_i(x_j) = \delta_{ij}$. Now for any $x \in X$ we have

$$x - \sum_j f_j(x)x_j \in \bigcap_i \ker f_i \subseteq \ker g,$$

And hence $g(x) = \sum_j f_j(x)g(x_j) = (\sum_j g(x_j)f_j)(x)$. Therefore $g = \sum_j g(x_j)f_j \in Y$. \square

Corollary 4.6. *Let X be a normed space. Then the norm and weak topologies have the same continuous linear functionals.*

Remark 4.7. Let X be a Banach space, and let X_0 be a proper dense linear manifold in X . We know that $X^* = X_0^*$, so that the $\sigma(X_0, X_0^*)$ -topology on X_0 is the restriction of the $\sigma(X, X^*)$ -topology. On the other hand, it turns out that X and X_0 induce different topologies on their common dual space. This is because of Proposition 4.5:

$$(X^*, \sigma(X^*, X))^* = X \supsetneq X_0 = (X^*, \sigma(X^*, X_0))^*.$$

For this reason, the next definition is made only for Banach spaces.

Definition 4.8. Let X be a Banach space. The $\sigma(X^*, X)$ -topology on X^* is called the *weak* topology*.

Remark 4.9. We note that the dual of a Banach space X carries both the weak* topology induced by X and the weak topology induced by X^{**} . The weak* topology is coarser than the weak topology on X^* . If X is reflexive, then evidently they coincide. But if X is not reflexive, then it follows from Proposition 4.5 that the weak* topology is strictly coarser than the weak topology.

Example 4.10. We let z denote the identity function on the circle \mathbb{T} .

- $z^n \rightarrow 0$ weak* in $L^\infty(\mathbb{T})$, by the Riemann-Lebesgue lemma (Theorem 2.7).
- $z^n \rightarrow 0$ weakly in $L^1(\mathbb{T})$, since $L^\infty \subseteq L^1$.
- $z^n \not\rightarrow 0$ in norm in $L^1(\mathbb{T})$ (exercise).
- $z^n \not\rightarrow 0$ weakly in $C(\mathbb{T})$, since evaluation at 1 is continuous on $C(\mathbb{T})$.

Now let e_n be the sequence that is 0 except for a 1 in the n th slot.

- $e_n \rightarrow 0$ weakly in c_0 .
- $e_n \not\rightarrow 0$ weakly in ℓ^1 .
- $e_n \rightarrow 0$ weak* in ℓ^1 .
- $e_n \rightarrow 0$ weak* in ℓ^∞ .
- $e_n \not\rightarrow 0$ in norm in any of the above spaces.

We now give some results on metrization and compactness in LCTVS's.

Theorem 4.11. *Let X be a LCTVS. The following are equivalent:*

- (1) *The topology of X is metrizable.*
- (2) *The topology of X is induced from a countable family of seminorms.*
- (3) *The topology of X has a countable neighborhood base at 0.*

Moreover, if the above hold then the metric can be chosen to be translation invariant.

Proof. (1) implies (3) since metric spaces are first countable. (3) implies (2) by Theorems 3.12 and 3.15, and Corollary 3.18 (details are left to the reader). To see that (2) implies (1), let p_1, p_2, \dots be seminorms inducing the topology of X . Define d on X by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{p_n(x - y)}{1 + p_n(x - y)}.$$

It is immediate that d is positive definite and symmetric. To see that the triangle inequality holds, note that the function $t/(1 + t)$ is increasing on $[0, \infty)$. Therefore if $f : X \rightarrow [0, \infty)$ is any function satisfying the triangle inequality,

$$\frac{f(x + y)}{1 + f(x + y)} \leq \frac{f(x) + f(y)}{1 + f(x) + f(y)} = \frac{f(x)}{1 + f(x) + f(y)} + \frac{f(y)}{1 + f(x) + f(y)} \leq \frac{f(x)}{1 + f(x)} + \frac{f(y)}{1 + f(y)}$$

Therefore d is a metric on X . It is easy to check that d defines the same topology on X as that induced by the seminorms p_n — we leave this as an exercise (that you really ought to do). Finally, note that the d is translation invariant by its construction. \square

Example 4.12. Let X be a locally compact Hausdorff space, and assume that X is σ -compact, i.e. that X is a countable union of compact subsets. Then in fact, there is a sequence of compact subsets whose interiors cover X . Namely, if $X = \cup_i K_i$ where K_i are compact, the shrinking lemma implies that each K_i has a compact neighborhood L_i . Then the interiors of the L_i 's cover the union of the K_i 's, and hence all of X . Now if K is any compact subset of X , finitely many L_i 's cover K . Thus there is n such that $p_K \leq \sum_{i=1}^n p_{L_i}$. Therefore the compact-open topology (Example 3.17) is induced by the countable family $\{p_{L_i} : i \in \mathbb{N}\}$. By Theorem 4.11, the compact-open topology is metrizable.

Proposition 4.13. *Let Y be a nondegenerate space of linear functionals on X . Then the $\sigma(X, Y)$ topology is metrizable if and only if Y has a countable basis.*

Proof. If Y has a countable basis f_1, f_2, \dots , then the $\sigma(X, Y)$ topology is induced from the seminorms $|f_1|, |f_2|, \dots$, and hence is metrizable by Theorem 4.11. If Y does not have a countable basis, we show that in the $\sigma(X, Y)$ topology there does not exist a countable neighborhood base at 0. A sequence of basic neighborhoods of 0 corresponds (essentially) to a sequence f_1, f_2, \dots in Y . We may as well assume that this sequence is linearly independent. (I.e. the sequence of neighborhoods may be taken to be $V(|f_1|, \dots, |f_n|; 1/n)$ for $n = 1, 2, 3, \dots$) Since Y does not have a countable basis, there is $g \in Y$ that is not in the span of the f_i . We claim that $V(|g|, 1)$ does not contain any of the above-mentioned basic neighborhoods. For if there is n such that $V(|f_1|, \dots, |f_n|; 1/n) \subseteq V(|g|, 1)$, then for any $t > 0$ we have $V(|f_1|, \dots, |f_n|; t/n) \subseteq V(|g|, t)$. It follows that $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$. As in the proof of Proposition 4.5, it follows that g is in the span of f_1, \dots, f_n , a contradiction. \square

Corollary 4.14. *If X is an infinite dimensional normed space, then the weak topology on X is not metrizable. If X is an infinite dimensional Banach space, then the weak* topology on X^* is not metrizable.*

Proof. Both of these facts rely on the fact that an infinite dimensional Banach space does not have a countable basis. The reason for this is that if it did, it would equal the union of a countable sequence of finite dimensional subspaces. These are closed by Corollary 1.15, and easily checked to be nowhere dense. This contradicts Baire's theorem. \square

Theorem 4.15. (*Alaoglu's theorem*) Let X be a Banach space, and let B denote the closed unit ball of X^* . Then B is compact in the weak* topology.

Proof. The proof is reminiscent of the proof of the Arzela-Ascoli theorem for locally compact Hausdorff spaces. Tychonoff's theorem is used in the same way here. For each $x \in X$ let $D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$, a compact subset of \mathbb{C} . Let $D = \prod_{x \in X} D_x$. Then D is compact for the product topology by Tychonoff's theorem. Define a function $e : B \rightarrow D$ by $e(f) = (f(x))_{x \in X}$ (e is for *evaluation*). First note that e is one-to-one. Next note that $e(B)$ is a closed subset of D , hence compact. To see this, let $e(f_a) \rightarrow z = (z_x)_{x \in X}$ in D . Then for any $x, y \in X$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} z_{\lambda x + y} &= \lim_a f_a(\lambda x + y) \\ &= \lim_a \lambda f_a(x) + f_a(y) \\ &= \lambda \cdot z_x + z_y, \end{aligned}$$

and

$$\begin{aligned} |z_x| &= \left| \lim_a f_a(x) \right| \\ &= \lim_a |f_a(x)| \\ &\leq \|x\|, \end{aligned}$$

since $f_a \in B$ for all a . Thus $x \mapsto z_x$ defines an element $g \in B$ such that $z = e(g)$. Finally, we note that $f_a \rightarrow f$ in B if and only if $e(f_a) \rightarrow e(f)$ in D , since this amounts to pointwise convergence in both places. Therefore $e : B \rightarrow e(B)$ is a homeomorphism. Since $e(B)$ is compact, so is B . \square

Theorem 4.16. Let X be a separable Banach space. Then the weak* topology on the unit ball of X^* is metrizable.

Proof. Let x_1, x_2, \dots be dense in X , and let $X_0 = \text{span}\{x_1, x_2, \dots\}$. Then the weak* topology is finer than the $\sigma(X^*, X_0)$ -topology. We claim that the two have equal restrictions to the closed unit ball, B , of X^* . For let $f_a, f \in B$, and suppose that $f_a \rightarrow f$ in the $\sigma(X^*, X_0)$ -topology. Let $y \in X$ and $\varepsilon > 0$. Choose i such that $\|y - x_i\| < \varepsilon/3$. Let a_0 be such that $a \succeq a_0$ implies that $|f_a(x_i) - f(x_i)| < \varepsilon/3$. Then for $a \succeq a_0$ we have

$$|f_a(y) - f(y)| \leq |f_a(y) - f_a(x_i)| + |f_a(x_i) - f(x_i)| + |f(x_i) - f(y)| \leq \|y - x_i\| + \varepsilon/3 + \|x_i - y\| < \varepsilon,$$

since $\|f_a\|, \|f\| \leq 1$. Therefore $f_a \rightarrow f$ weak*. \square

Example 4.17. (Invariant means) Letting \mathbb{T} have normalized Lebesgue measure, we define a linear functional on $L^\infty(\mathbb{T})$ by $\varphi(f) = \int_{\mathbb{T}} f$. This has the properties:

- (1) (*positivity*) If $f \geq 0$ then $\varphi(f) \geq 0$.
- (2) $\|\varphi\| = 1$.
- (3) (*translation invariance*) For any $f \in L^\infty(\mathbb{T})$ and $z \in \mathbb{T}$, $\varphi(U_z f) = \varphi(f)$, where $U_z f(w) = f(z^{-1}w)$.

On non-compact groups, such as \mathbb{R} or \mathbb{Z} , Lebesgue measure is infinite, and we can't play this trick. Nevertheless, there exist such linear functionals in these contexts too, courtesy of Alaoglu. Here is a sketch of the proof for \mathbb{R} .

For $n \in \mathbb{N}$ let $\varphi_n \in L^\infty(\mathbb{R})^*$ be defined by

$$\varphi_n(f) = \frac{1}{2n} \int_{-n}^n f.$$

Then $\|\varphi_n\| = 1$, for all n . Since the unit ball of $L^\infty(\mathbb{R})^*$ is weak*-compact, by Theorem 4.15, the set $\{\varphi_n : n \in \mathbb{N}\}$ has a weak* limit point, φ , in the unit ball of $L^\infty(\mathbb{R})^*$. Since $\varphi_n(1) = 1$ and φ_n is positive, it is easy to see that φ has norm one and is positive. To see that φ is translation invariant, fix $f \in L^\infty(\mathbb{R})$ and $t \in \mathbb{R}$. Note that

$$\begin{aligned} |\varphi_n(U_t f) - \varphi_n(f)| &= \frac{1}{2n} \left| \int_{-n-t}^{n-t} f - \int_{-n}^n f \right| \\ &= \frac{1}{2n} \left| \int_{-n-t}^{-n} f - \int_{n-t}^n f \right| \\ &\leq \frac{1}{2n} \cdot 2t \cdot \|f\|_\infty \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Let $\varepsilon > 0$. Choose n_0 so that $\frac{t}{n_0} \|f\|_\infty < \varepsilon/3$. Then φ is still in the weak* closure of $\{\varphi_n : n \geq n_0\}$. Then there is $n \geq n_0$ such that $|\varphi(U_t f) - \varphi_n(U_t f)| < \varepsilon/3$ and $|\varphi(f) - \varphi_n(f)| < \varepsilon/3$. Then

$$|\varphi(U_t f) - \varphi(f)| \leq |\varphi(U_t f) - \varphi_n(U_t f)| + |\varphi_n(U_t f) - \varphi_n(f)| + |\varphi_n(f) - \varphi(f)| < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, we have $\varphi(U_t f) = \varphi(f)$.

5. SEPARATION THEOREMS

Definition 5.1. Let X be a LCTVS. A *(closed) hyperplane* in X is a set of the form $\{f = c\}$, where f is a real-valued continuous linear functional on X (and $c \in \mathbb{R}$). A *closed* (respectively, *open*) *halfspace* is a set of the form $\{f \leq c\}$ (respectively, $\{f < c\}$), with f and c as before. Subsets E and F of X are *separated* (by a closed hyperplane) if there are f and c such that $f \leq c$ on E , and $f \geq c$ on F . E and F are *strictly separated* if they are separated, and the associated inequalities are strict.

Theorem 5.2. Let X be a LCTVS, and let $E, F \subseteq X$ be disjoint convex subsets.

- (1) If E is open then E and F are separated.
- (2) If E and F are both open then E and F are strictly separated.
- (3) If E is compact and F is closed, then E and F are strictly separated.

Proof. (1): Note that $E - F$ is open and convex. Choose $x_0 \in E - F$, and let $W = E - F - x_0$, a convex open neighborhood of 0. Let p_W be the gauge functional of W . Recall from the proof of Theorem 3.11 that p_W is sublinear (balancedness of U was only required for homogeneity for non-positive scalars). Moreover, in the proof that $\{p_W < 1\} \subseteq W$, we only needed $\lambda W \subseteq W$ for $\lambda \in [0, 1]$ — this follows in the present case from convexity, and the fact that $0 \in W$.

Now, since $E \cap F = \emptyset$, $-x_0 \notin W$. Hence $p_W(-x_0) \geq 1$. Define a linear functional on $\mathbb{R}x_0$ by $f_0(\lambda x_0) = -\lambda$. We claim that $f_0 \leq p_W|_{\mathbb{R}x_0}$. For, if $\lambda \geq 0$ then $f_0(\lambda x_0) = -\lambda \leq 0 \leq p_W(\lambda x_0)$, while if $\lambda < 0$ then $f_0(\lambda x_0) = -\lambda \leq -\lambda \cdot p_W(-x_0) = p_W(\lambda x_0)$. Thus the Hahn-Banach theorem provides a linear functional $f : X \rightarrow \mathbb{R}$ that extends f_0 , and satisfies $f \leq p_W$.

Let $U \subseteq W$ be a balanced convex open set, and let p_U be the gauge of U . Then p_U is a seminorm. From the definition of the gauge functional we have that $p_W \leq p_U$, and hence for $x \in X$,

$$|f(x)| = \pm f(x) = f(\pm x) \leq p_W(\pm x) \leq p_U(\pm x) = p_U(x).$$

Therefore $|f|$ is continuous (by Theorem 3.15(3)), and hence f is continuous.

Recall that $f(x_0) = -1$. For any $x \in E$ and $y \in F$ we have $x - y - x_0 \in W$, so $f(x - y - x_0) \leq p_W(x - y - x_0) \leq 1$ (proof of Theorem 3.11). Therefore $f(x) \leq f(y)$. We may let $c = \sup_{x \in E} f(x)$.

(2): Note that in (1), $f(E)$ is open in \mathbb{R} since f is continuous and E is open. Thus the inequality $f \leq c$ on E must actually be a strict inequality. If F is also open then both inequalities must be strict.

(3): We claim that there is a balanced convex open set V such that $(E + V) \cap F = \emptyset$. For suppose not. Let \mathcal{A} be the family of all non-empty balanced convex open sets, directed by reverse inclusion. For $V \in \mathcal{A}$ choose $x_V \in E$ and $y_V \in (x_V + V) \cap F$. Since E is compact, there is a subnet $(x_{V_b})_{b \in B}$ with $x_{V_b} \rightarrow x \in E$. Choose a balanced convex open set U such that $(x + U) \cap F = \emptyset$ (this is possible since F is closed). Choose $b \in B$ such that $x_{V_b} \in x + \frac{1}{2}U$ and $V_b \subseteq \frac{1}{2}U$. Then

$$y_{V_b} \in x_{V_b} + V_b \subseteq (x + \frac{1}{2}U) + \frac{1}{2}U \subseteq x + U,$$

a contradiction.

Now, with $V \in \mathcal{A}$ such that $(E + V) \cap F = \emptyset$, we have that $(E + \frac{1}{2}V) \cap (F + \frac{1}{2}V) = \emptyset$. The result now follows from part (2). \square

Corollary 5.3. *In a LCTVS, a closed convex set equals the intersection of all (closed) halfspaces that contain it.*

Corollary 5.4. *The norm and weak topologies on a Banach space have the same closed convex sets.*

Corollary 5.5. *Let $x_n \rightarrow x$ weakly in a Banach space. Then there is a sequence (y_n) in $\text{co}\{x_1, x_2, \dots\}$ such that $y_n \rightarrow x$ in norm.*

Proof. Let $A = \{x_1, x_2, \dots\}$. Then x is in the weak closure of A , hence x is in the weak closure of $\text{co}(A)$. By Corollary 5.4, this equals the norm closure of $\text{co}(A)$. \square

Remark 5.6. Let X be a Banach space, and let $E \subseteq X$ be convex. We let \overline{B}_r denote the closed ball of radius r in X , and \overline{C}_r the closed ball of radius r in X^* . It is easy to see that E is norm closed if and only if $E \cap \overline{B}_r$ is norm closed for all $r > 0$. By Corollary 5.4 we see that E is weakly closed if and only if $E \cap \overline{B}_r$ is weakly closed for all $r > 0$. (Exercise: why do we need to use Corollary 5.4?)

The same result is true for the weak* topology also, but it is deeper. We'll give the statement, but not the proof.

Theorem 5.7. *(Krein-Smulian) With notation as above, E is weak* closed if and only if $E \cap \overline{C}_r$ is weak* closed for all $r > 0$.*

Definition 5.8. Let X be a vector space, and let $E \subseteq X$ be a convex subset. A point $x \in E$ is an *extreme point* if it is impossible to write $x = (y + z)/2$ with $y, z \in E$ and $y \neq z$.

Exercise 5.9. x is an extreme point of E if and only if it is impossible to write $x = (1-t)y + tz$ with $y, z \in E$, $y \neq z$, and $0 < t < 1$.

Example 5.10. (1) \mathbb{D} has no extreme points.

(2) $\overline{\mathbb{D}}$ has \mathbb{T} as its set of extreme points.

(3) $\{f \in L^1[0, 1] : \|f\|_1 \leq 1\}$ has no extreme points.

(4) Let $1 < p < \infty$. $\{f \in L^p[0, 1] : \|f\|_p \leq 1\}$ has $\{f \in L^p[0, 1] : \|f\|_p = 1\}$ as its set of extreme points.

Definition 5.11. Let X be a LCTVS, and let $A \subseteq X$. The *closed convex hull* of A is the set

$$\overline{\text{co}}(A) = \bigcap \{E : A \subseteq E, E \text{ is closed and convex}\}.$$

Remark 5.12. It is an easy exercise to check that $\overline{\text{co}}(A) = \overline{\text{co}(A)}$.

Theorem 5.13. (*Krein-Milman theorem*) Let E be a compact convex subset of a LCTVS. Then E equals the closed convex hull of its set of extreme points.

Corollary 5.14. The set of extreme points of a compact convex subset of a LCTVS is non-empty.

Proof. (of Theorem 5.13) A subset $F \subseteq E$ will be called an *extreme set* if it is non-empty, compact, convex, and has the property that if $x_1, x_2 \in E$ are such that $(x_1 + x_2)/2 \in F$ then $x_1, x_2 \in F$. Thus the extreme points of E are the singleton extreme sets. Note that if $F_1 \supseteq F_2 \supseteq \dots$ are extreme sets, then $\bigcap_i F_i$ is an extreme set (it is non-empty by the finite intersection property). More generally, if $\{F_i\}_{i \in I}$ is a totally ordered (by inclusion) family of extreme sets, then $\bigcap_{i \in I} F_i$ is also an extreme set. Since E is an extreme set, the family of extreme sets is non-empty. Therefore Zorn's lemma implies that there is a minimal extreme set.

We claim that a minimal extreme set must be a singleton. For suppose not. Then let K be a minimal extreme set that has more than one point. By the Hahn-Banach theorem there is a continuous real-valued linear functional f that is non-constant on K . Let $c = \min_K f$, and put $K_0 = K \cap f^{-1}(c)$. We will check that K_0 is an extreme set. Let $x \in K_0$, $x = (x_1 + x_2)/2$ with $x_1, x_2 \in E$. Since $x \in K$ and K is extreme, then $x_1, x_2 \in K$. Then

$$c = f(x) = \frac{1}{2}(f(x_1) + f(x_2)) \geq \frac{1}{2}(c + c) = c.$$

Since $x_1, x_2 \in K$ we know that $f(x_1), f(x_2) \geq c$. Thus it follows that $f(x_1) = f(x_2) = c$, so that $x_1, x_2 \in K_0$. Hence K_0 is an extreme set. Since K is a minimal extreme set, $K_0 = K$, contradicting the assumption that f is non-constant on K .

Thus we have proved that E has at least one extreme point. Let E_0 be the closed convex hull of the extreme points of E . We claim that $E_0 = E$. To see this, suppose that there is a point $x \in E \setminus E_0$. By Theorem 5.2(3) there is a continuous real-valued linear functional g on X , and $a \in \mathbb{R}$, such that $g|_{E_0} > a$ and $g(x) < a$. Let $b = \min_E g$. Then $b = g(y)$ for some $y \in E$, and $y \notin E_0$. Let $F = E \cap g^{-1}(b)$. F is compact, convex, nonempty, and extreme (by the same proof as before). By what we have already proved, F must have an extreme point, and this must also be an extreme point of E (since F is an extreme set in E). This contradiction finishes the proof. \square

Example 5.15. Let T be a compact metric space, and let $M(T)$ be the complex regular Borel measures on T . Let $X \subseteq M(T)$ be the set of regular Borel probability measures on T . Then X is convex and weak* closed, hence compact by Alaoglu's theorem.

Let $\varphi : T \rightarrow T$ be a continuous map. Call a measure $\mu \in M(T)$ φ -invariant if $\mu(\varphi^{-1}(E)) = \mu(E)$ for all Borel sets $E \subseteq T$ (equivalently, if $\int f d\mu = \int f \circ \varphi d\mu$ for all $f \in C(T)$). Let X_φ be the set of φ -invariant probability measures. X_φ is a weak* closed convex set. We claim that X_φ is non-empty. This is proved in the same way that we proved the existence of invariant means on \mathbb{R} (Example 4.17): let μ_0 be any probability measure, and let $\mu_n = \frac{1}{n}(\mu_0 + \mu_0 \circ \varphi^{-1} + \cdots + \mu_0 \circ (\varphi^{-1})^{n-1})$. Let μ be a weak* cluster point of the set $\{\mu_n : n \in \mathbb{N}\}$, and check that μ is invariant.

Now, the Krein-Milman theorem implies that X_φ has extreme points.

Definition 5.16. An invariant measure μ is called *ergodic* if whenever $E \subseteq T$ is Borel and invariant (i.e. $\varphi^{-1}(E) = E$), then $\mu(E) = 0$ or 1.

Exercise 5.17. $\mu \in X_\varphi$ is ergodic if and only if μ is an extreme point of X_φ .

6. COMPLEMENTED SUBSPACES

Definition 6.1. Let X be a TVS.

- (1) An operator $P \in L(X)$ is an *idempotent* if $P^2 = P$.
- (2) A subspace $M \subseteq X$ is *complemented* if there is a subspace $N \subseteq X$ such that $M \cap N = \{0\}$ and $M + N = X$. In this case we call M and N *complementary subspaces*.

Theorem 6.2. Let X be a Banach space.

- (1) If $P \in B(X)$ is an idempotent then $\ker P$ and PX are complementary subspaces.
- (2) If M and N are complementary subspaces of X then there exists a unique idempotent $P \in B(X)$ such that $N = \ker P$ and $M = PX$.

Proof. (1) We first note that $(I - P)P = P - P^2 = P - P = 0$. Now, if $y = Px$ for some $x \in X$ then $(I - P)y = (I - P)Px = 0$. Therefore $PX \subseteq \ker(I - P)$. Conversely, if $y \in \ker(I - P)$ then $y - Py = (I - P)y = 0$, so $y = Py \in PX$. Thus $PX = \ker(I - P)$ is closed. Of course $\ker P$ is also closed. Let $z \in PX \cap \ker P$. Then $z = Px$ for some $x \in X$, so $z = P^2x = Pz = 0$. Thus $PX \cap \ker P = \{0\}$. Let $w \in X$. Then $w = (P + (I - P))w = Pw + (I - P)w \in PX + \ker P$. Therefore $X = PX + \ker P$.

(2) Give $M \oplus N$ the norm $\|(y, z)\| = \|y\| + \|z\|$. Then $M \oplus N$ is a Banach space, since M and N are separately Banach spaces. Define a map $T : M \oplus N \rightarrow X$ by $(y, z) \mapsto y + z$. Then T is one-to-one and onto, hence is an isomorphism by the inverse mapping theorem. Now define $P \in B(X)$ by $P = \pi_1 \circ T^{-1}$, where $\pi_1 : M \oplus N \rightarrow M$ is the projection. It's easy to see that P is an idempotent with $M = PX$ and $N = \ker P$. For the uniqueness, let $Q \in B(X)$ be an idempotent with $M = QX$ and $N = \ker Q$. Then for any $x \in X$ we have $QPx = Px$ and $Q(I - P)x = 0$. Hence $Qx = Q(Px + (I - P)x) = QPx = Px$. \square

Theorem 6.3. Let X be a Banach space, and let $Y \subseteq X$ be a subspace. If $\dim Y < \infty$ or $\text{codim } Y < \infty$ then Y is complemented.

Proof. When $\dim Y < \infty$, this is homework problem 27. Now assume that $\text{codim } Y = n < \infty$. Let $x_1, \dots, x_n \in X$ be such that $\{x_i + Y : 1 \leq i \leq n\}$ is a basis for X/Y . Define $T \in B(X/Y, X)$ by $T(x_i + Y) = x_i$, and extending by linearity. If $\pi : X \rightarrow X/Y$ is the projection, then $\pi \circ T = \text{id}_{X/Y}$ (as can be seen by checking on the basis for X/Y). Let $P = T \circ \pi \in B(X)$. Then $P^2 = T \circ \pi \circ T \circ \pi = T \circ \pi = P$, so P is an idempotent. Since T is one-to-one, we have $\ker P = \ker \pi = Y$. It now follows from Theorem 6.2(1) that Y is complemented. \square

Remark 6.4. We should point out that part of the hypotheses of Theorem 6.3 is the assumption that Y is a subspace of X , i.e. that Y is closed. This is automatic if $\dim Y < \infty$, but not if $\operatorname{codim} Y < \infty$.