Solve three problems from among these and past unsolved problems.

**38.** Let $X$ be an infinite dimensional Banach space. Let $B = \{x \in X : \|x\| \leq 1\}$ be the closed unit ball, and let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere.

(i) Prove that $B$ is weakly closed.
(ii) Prove that $S$ is not weakly closed.
(iii) Find (with proof) the weak closure of $S$.

**39.** For $1 \leq m < n$ let $x_{mn}$ be the sequence

$$x_{mn}(j) = \begin{cases} 1, & \text{if } j = m \\ m, & \text{if } j = n \\ 0, & \text{otherwise.} \end{cases}$$

Let $E = \{x_{mn} \mid 1 \leq m < n\} \subseteq \ell^2$. Prove that 0 is in the weak closure of $E$, but that no sequence in $E$ converges weakly to 0.

**40.** A Banach space is called *strictly convex* if every point of the unit sphere is an extreme point of the unit ball.

(i) Prove that a Banach space is strictly convex if and only if $\|x + y\| < \|x\| + \|y\|$ for every pair of linearly independent elements $x$ and $y$.
(ii) Prove that $L^p(\mu)$ is strictly convex if $1 < p < \infty$.
(iii) Describe (with proof) all extreme points of the unit ball of $L^\infty(\mu)$.

**41.** Let $1 < p < \infty$, and let $E \subseteq L^p(\mu)$ be a closed convex set. Prove that $E$ contains a unique element of smallest norm.