Solve three problems from among these and past unsolved problems.

34. Let X and Y be Banach spaces, and let $A \subseteq L(X,Y)$. Suppose that $\sup\{|g(Tx)| : T \in A\} < \infty$ for each $x \in X$ and each $g \in Y^*$. Prove that A is bounded.

35. Let X and Y be normed spaces, and let $\eta : X \times Y \to \mathbf{C}$ be a bilinear functional. Suppose that η is *separately continuous*: for fixed $x \in X$, $\eta(x,y)$ is a continuous functional of y, and for fixed $y \in Y$, $\eta(x,y)$ is a continuous functional of x. Assume that at least one of X and Y is a Banach space. Prove that η is bounded (i.e. that there isa constant C such that $|\eta(x,y)| \leq C\|x\| \cdot \|y\|$).

36. Let X be a TVS.
(i) Let $E \subseteq X$ be convex. Prove that the interior of E is convex.
(ii) Let $F \subseteq X$ be closed. Prove that F is convex if and only if $(x+y)/2 \in F$ whenever $x, y \in F$.

37. Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. Let X be the vector space of all measurable complex-valued functions on Ω (where functions are identified if they are equal almost everywhere). For $f \in X$ define

$$\lambda(f) = \int \frac{|f|}{1+|f|}\, d\mu.$$

For $f, g \in X$ define $d(f,g) = \lambda(f-g)$.
(i) Prove that $d(f_i, f) \to 0$ if and only if $f_i \to f$ in μ-measure.
(ii) Prove that d is a complete metric, and that (X,d) is a TVS.
(iii) Let $\Omega = [0,1]$ and let μ be Lebesgue measure. Prove that $X^* = \{0\}$. (Hint: Suppose that $\phi \in X^*$ with $\phi \neq 0$. Prove that there exist sets S of arbitrarily small measure for which $\phi(\chi_S) \neq 0$.)