MAT 571 **HW 7** **Due Thursday, 3/8/07**

Solve three problems from among these and past unsolved problems.

27. Let X be a normed space, and let $Y \subseteq X$ be a finite dimensional subspace. Prove that there exists a subspace Z of X such that $Y \cap Z = \{0\}$ and $Y + Z = X$.

28. Let X be a normed space, and let $Y \subseteq X$ be a subspace. Let $\pi : X \to X/Y$ be the quotient map, and let $\pi^* : (X/Y)^* \to X^*$ be its adjoint.

(i) Prove that $\pi^*\big((X/Y)^*\big) = Y^\perp$.
(ii) Prove that π^* is an isometry.

(Thus we say that $(X/Y)^*$ '=' $Y^\perp$.)

29. Let X be a normed space, and suppose that X^* is separable (i.e. has a countable dense subset). Prove that X is separable. (Hints: Let $\{f_1, f_2, \ldots\}$ be dense in the unit sphere of X^*. Choose $\{x_1, x_2, \ldots\}$ in the unit sphere of X such that $\big|f_n(x_n)\big| > 1/2$. Let Y be the closed linear span of $\{x_1, x_2, \ldots\}$. If $Y \neq X$ use the Hahn-Banach theorem to obtain a contradiction.)

30. Let X be a Banach space, and let $Y \subseteq X$ be a subspace. Suppose that Y and X/Y are reflexive. Prove that X is reflexive.

31. Let X and Y be Banach spaces, and let $T \in B(X, Y)$. Prove that T is bounded below if and only if T is one-to-one and has closed range. (Recall that a linear map T between normed spaces is called *bounded below* if there is a positive constant c such that $\|Tx\| \geq c\|x\|$ for all x.)

32. Let Y and Z be subspaces of a Banach space X such that $Y \cap Z = \{0\}$. Prove that $Y + Z$ is closed if and only if $\inf\big\{\|y + z\| \ : \ y \in Y,\ z \in Z,\ \|y\| = \|z\| = 1\big\} > 0$.

33. Let X be a Banach space, and let $E : X \to X$ be a linear map. Suppose that E is *idempotent* (i.e. suppose that $E^2 = E$). Prove that E is continuous if and only if the kernel and range of E are closed.