Solve three problems from among these and past unsolved problems.

9. Let X be a topological space. Prove that X is Hausdorff if and only if every convergent net in X has a unique cluster point. (Note that a limit of a net is also a cluster point of the net.)

10. Let X have the weak topology induced by a family $\mathcal{F}$ of functions on X. Let $x \in X$ and let $(x_a)_{a\in A}$ be a net in X. Prove that $x_a \to x$ if and only if $f(x_a) \to f(x)$ for each $f \in \mathcal{F}$.

11. Let X be a set, and let $f : X \to \mathbf{C}$ be a function. Let $\mathcal{F}$ be the collection of all nonempty finite subsets of X, directed by inclusion. For $A \in \mathcal{F}$ let $z_A = \sum_{x\in A} f(x)$. Prove that the net $(z_A)_{A\in\mathcal{F}}$ converges in $\mathbf{C}$ if and only if the following two conditions are satisfied:

(i) $\{x : f(x) \neq 0\}$ is countable.
(ii) Writing $\{x : f(x) \neq 0\} = \{x_1, x_2, x_3, \ldots\}$, then $\sum_{n=1}^{\cdot} \big|f(x_n)\big| < \infty$.

In this case, $\lim_{A\in\mathcal{F}} z_A = \sum_{n=1}^{\infty} f(x_n)$.

12. Let $(X, \mathcal{T})$ be a compact Hausdorff space, and let $\mathcal{T}'$ be another topology on X.

(i) Prove that if $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ then $(X, \mathcal{T}')$ is Hausdorff but not compact.
(ii) Prove that if $\mathcal{T}'$ is strictly coarser than $\mathcal{T}$ then $(X, \mathcal{T}')$ is compact but not Hausdorff.

13. Recall that each number in $[0, 1)$ has a base-2 expansion as a(n infinite) sum of inverse powers of 2, with coefficients in $\{0, 1\}$. The expansion is unique unless the number is a dyadic rational (i.e. a rational whose denominator is a power of 2). If we adopt the convention that for a dyadic rational we choose the finite expansion, then the expansion is unique. Then the nth coefficient defines a function $a_n : [0, 1) \to \{0, 1\}$. The functions a_1, a_2, ... can be characterized by the equation

$$x = \sum_{n=1}^{\infty} a_n(x)2^{-n}, \; x \in [0, 1)$$

(together with the restriction that for no x is the sequence $a_n(x)$ eventually equal to 1). Prove that the sequence (a_n) in the space $\{0, 1\}^{[0,1)}$ has no pointwise-convergent subsequence. (Thus the product topology on $\{0, 1\}^{[0,1)}$ is not sequentially compact.)

14. Let X be a topological space.

(i) Prove that if X is sequentially compact then it is countably compact.
(ii) Prove that X is countably compact if and only if every sequence in X has a cluster point.
(iii) Prove that if X is countably compact and first countable then it is sequentially compact.
(iv) Suppose that X is normal. Prove that X is countably compact if and only if $C(X) = C_b(X)$. (Hint: use part (ii) and Tietze's extension theorem.)