

Solve three problems from among these and past unsolved problems.

**49.** (Recall the notation for translation of functions on  $\mathbf{R}$ :  $(T_y f)(x) = f(x - y)$ , whenever  $f : \mathbf{R} \rightarrow \mathbf{C}$  and  $x, y \in \mathbf{R}$ .) Let  $f \in L^p(\mathbf{R})$ . If there exists  $h \in L^p(\mathbf{R})$  such that

$$\lim_{t \rightarrow 0} \left\| \frac{1}{t} (T_{-t} f - f) - h \right\|_p = 0,$$

then  $h$  is called the *strong  $L^p$  derivative* of  $f$ .

Let  $p$  and  $q$  be conjugate exponents ( $1 \leq p < \infty$ ), let  $f \in L^p$  have strong  $L^p$  derivative  $h$ , and let  $g \in L^q$ . Prove that  $f * g$  is a differentiable function, and that  $(f * g)' = h * g$ .

**50.** Let  $f \in L^p$  ( $1 \leq p < \infty$ ), and suppose that  $f$  has strong  $L^p$  derivative  $h$ . Prove that (possibly after modification on a null set)  $f$  is absolutely continuous on every bounded interval, and  $f' = h$ . (Hint: let  $g \in C_c(\mathbf{R})$  with  $\int g = 1$ . Then  $f * g_t \rightarrow f$  in  $L^p$ , and by the previous problem  $(f * g_t)' \rightarrow h$  in  $L^p$ .)

**51.** Suppose that  $f : \mathbf{R} \rightarrow \mathbf{C}$  is absolutely continuous on every bounded interval, and that  $f$  and  $f'$  are both in  $L^p$  ( $1 \leq p < \infty$ ). Prove that  $f'$  is the strong  $L^p$  derivative of  $f$ . (Hint: write

$$\frac{f(x+t) - f(x)}{t} - f'(x) = \frac{1}{t} \int (f'(x+y) - f'(x)) \chi_{[0,t]}(y) \cdot \chi_{[0,t]}(y) dy,$$

and use Hölder's inequality as in the proof from class of Young's inequality.)

**52.** (Qualitative uncertainty principle)

- (i) Prove that if  $f \in L^1(\mathbf{R})$  and  $f * f = 0$  a.e., then  $f = 0$  a.e.
- (ii) Let  $f \in L^1$ , and assume that both  $f$  and  $\widehat{f}$  have compact support. Prove that  $f = 0$  a.e. (Hint: Choose  $N > 0$  such that both  $f$  and  $\widehat{f}$  are supported in  $(-N, N)$ . Show that  $(C_N \cdot M^k f) * f = 0$  a.e. for every integer  $k \geq 0$ . Use the Weierstrass approximation theorem to approximate  $C_N$  by a polynomial on  $[-N, N]$ . (We use the notation  $C_a(x) = \exp(2\pi i a x)$ .)

**53.** Let  $f_k$  be the characteristic function of the interval  $(-k, k)$ , and let  $g_k = f_k * f_1$ .

- (i) Find explicitly the Fourier transform of  $f_k$ .
- (ii) Find  $g_k$  explicitly, and find  $\|g_k\|_\infty$ .
- (iii) Prove that  $\|\check{g}_k\|_1 \rightarrow \infty$  as  $k \rightarrow \infty$ . (Hint: Use the estimate  $\sin 2\pi y \geq \pi y$  for  $0 \leq y \leq \frac{1}{2\pi}$  to get a lower bound for  $\int_0^{1/2\pi} |\check{g}_k|$ .)
- (iv) Prove that the Fourier transform  $\mathcal{F} : L^1(\mathbf{R}) \rightarrow C_0(\mathbf{R})$  is not bounded below. (A linear map  $T$  between normed spaces is called *bounded below* if there is a constant  $c > 0$  such that  $\|T(\xi)\| \geq c\|\xi\|$  for all  $\xi$ .)