Mat 300, Spielberg **Final Exam Review Problems** **Spring 07**

1. Show that the formulas $p \to (q \wedge r)$ and $(p \to q) \wedge (p \to r)$ are logically equivalent. Do this by using known equivalences (such as commutativity, associativity, distributivity, and DeMorgan's laws) to modify one of the formulas until it becomes the other. Show all steps in the process.

2. Let A, B and C be sets.

(i) One of $(A \setminus B) \cup C$ and $A \setminus (B \cup C)$ must be a subset of the other. Which one is it? Prove your answer.
(ii) Prove that the opposite containment to the one you proved in part (i) does not hold in general.

3. Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be indexed families of sets. Suppose that for all $i \in I$ and for all $j \in J$ we have $A_i \subseteq B_j$. Prove that

$$\bigcup_{i \in I} A_i \subseteq \bigcap_{j \in J} B_j.$$

4. Express each of the following sets in two ways: in the form $\{x \in A : P(x)\}$, and in the form $\{F(y) : y \in B\}$ (where A and B are sets, $P(x)$ is a statement about the variable x, and $F(y)$ is an expression involving the variable y).

(i) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\}$
(ii) $\{1, 2, 4, 8, 16, \cdots\}$
(iii) $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \cdots\}$

5. Let $x \in \mathbf{R}$. Suppose that $|x + 6| < 3$. Prove that $x < -3$.

6. Let $x \in \mathbf{R}$. Suppose that $|x + 3| > 6$. Prove that $x^2 > 9$.

7. Prove that

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n},$$

for all $n \in \mathbf{N}$ with $n \geq 2$.

8. Prove that for all $n \in \mathbf{N}$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

9. Prove that for all $n \geq 0$,

$$\sum_{k=0}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

10. Let A, B and C be sets. Prove that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

11. Define $f : [0, \infty) \to [0, \infty)$ by

$$f(x) = \frac{x^2}{x+1}.$$

Prove that f is one-to-one and onto.

12. Let $f : A \to B$ be onto. Let $T \subseteq B$. Prove that $f\big(f^{-1}(T)\big) = T$.

13. Let $f : A \to B$.

(i) Let $T_1, T_2 \subseteq B$. Prove that $f^{-1}(T_1 \setminus T_2) = f^{-1}(T_1) \setminus f^{-1}(T_2)$.
(ii) Let $S_1, S_2 \subseteq A$. Prove that $f(S_1 \setminus S_2) \supseteq f(S_1) \setminus f(S_2)$.

14. Let $f : A \to B$ and $g : B \to C$ be functions.

(i) Suppose that $g \circ f$ is onto and that g is one-to-one. Prove that f is onto.
(ii) Suppose that f is not onto and that g is one-to-one. Prove that $g \circ f$ is not onto.

15. Let A be a set, and let $f : A \to \mathbf{N}$ be a one-to-one function. Prove that A is countable.

16. The relation R on $(0, \infty)$ is defined by: xRy if there exists $k \in \mathbf{Z}$ such that $y = x^k$. Is R reflexive? symmetric? transitive? Prove your answers.