MAT 571 **HW 5** **Due Thursday, 2/23/06**

Solve three problems. from among these and past unsolved problems.

17. A continuous function f is called *proper* if $f^{-1}(K)$ is compact whenever K is compact. Let $f : X \to Y$ be a continuous map between LCH spaces. Prove that f is proper if and only if f extends to a continuous map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ by setting $\widetilde{f}(\infty_X) = \infty_Y$.

18. Prove that every open subset of a second countable LCH space is σ-compact.

19. Let X be a σ-compact LCH space, and let U_1, U_2, ... be open subsets with compact closures such that $\overline{U_n} \subseteq U_{n+1}$ and $X = \cup_n U_n$. For $f \in C(X)$, and m, $n \in \mathbf{N}$, recall that

$$V(f, \overline{U_n}, m) = \{g \in C(X) : \|f - g\|_{\overline{U_n}} < \frac{1}{m}\},$$

where $\|h\|_K$ indicates the supremum of $|h|$ over the compact set K. Prove that

$$\{V(f, \overline{U_n}, m) : m, n \in \mathbf{N}\}$$

is a neighborhood base at f for the topology of uniform convergence on compact subsets. (First prove it for the case $f = 0$.)

20. Let $\{X_i\}_{i \in I}$ be a family of non-empty Hausdorff topological spaces such that X_i is non-compact for infinitely many i. Prove that every compact subset of $\prod_{i \in I} X_i$ is nowhere dense. (A subset E of a topological space is *nowhere dense* if $\operatorname{int}(\overline{E}) = \emptyset$.)

21. Is the result of problem 20 still true without the assumption that the X_i are Hausdorff? Prove, or give a counter-example.