

Solve three problems. from among these and past unsolved problems.

5. Let X be a set, let \mathcal{F} be a collection of real-valued functions on X , and let \mathcal{T} be the weak topology defined by \mathcal{F} . Prove that \mathcal{T} is Hausdorff if and only if for all $x \neq y$ in X there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.
6. Let (X, \mathcal{T}) be a completely regular topological space, and let $C(X)$ denote the set of all continuous real-valued functions on X . Prove that \mathcal{T} is the weak topology induced by $C(X)$.
7. Let X be a topological space, let \sim be an equivalence relation on X , let \bar{X} be the quotient space (the space of equivalence classes), and let $\pi : X \rightarrow \bar{X}$ denote the quotient map ($\pi(x) = [x]$). Define $\mathcal{T} = \{U \subseteq \bar{X} : \pi^{-1}(U) \text{ is open in } X\}$. (Some useful terminology: a subset of X is called *saturated* if it is a union of equivalence classes. Thus $E \subseteq X$ is saturated if and only if $E = \pi^{-1}(\pi(E))$. If we identify \bar{X} with the collection of equivalence classes, then we may identify \mathcal{T} with the collection of open saturated subsets of X .) Prove the following:
- (i) \mathcal{T} is a topology on \bar{X} (called the *quotient topology*).
 - (ii) If Y is a topological space, and $f : \bar{X} \rightarrow Y$ is a function, then f is continuous if and only if $f \circ \pi$ is continuous.
 - (iii) \bar{X} is T_1 if and only if every equivalence class is closed.
8. Let G be the following group of 2×2 matrices:

$$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbf{R} \right\}.$$

Define an equivalence relation on \mathbf{R}^2 by $x \sim y$ if $Ax = y$ for some $A \in G$. Give a simple description of the quotient space, and give (with proof) a convenient base for the quotient topology. Prove that the quotient topology is T_1 but not Hausdorff.