**MAT 571** **HW 10** **Due Thursday, 4/6/06**

Solve three problems. from among these and past unsolved problems.

**39.** Let $M$ and $N$ be closed subspaces of a Banach space, with $M \cap N = \{0\}$. Let $M + N = \{x + y : x \in M,\ y \in N\}$. Prove that $M + N$ is closed if and only if

$$\inf\big\{\|x + y\| : x \in M,\ y \in N,\ \|x\| = \|y\| = 1\big\} > 0.$$

Hint: Let $Z = M \oplus N$, $W = \overline{M + N}$, and $T : Z \to W$ given by $T(x, y) = x + y$.

**40.** Let $X$ and $Y$ be normed spaces, and let $\eta : X \times Y \to \mathbf{C}$ be a bilinear functional. Suppose that $\eta$ is *separately continuous*: for fixed $x \in X$, $\eta(x, y)$ is a continuous functional of $y$, and for fixed $y \in Y$, $\eta(x, y)$ is a continuous functional of $x$. Assume that at least one of $X$ and $Y$ is a Banach space. Prove that $\eta$ is bounded (i.e. that there is a constant $C$ such that $\big|\eta(x, y)\big| \le C\|x\| \cdot \|y\|$).

**41.** Let $X$ be an infinite dimensional Banach space. Let $B = \{x \in X : \|x\| \le 1\}$ be the closed unit ball, and let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere.

(i) Prove that $B$ is weakly closed.
(ii) Prove that $S$ is not weakly closed.
(iii) Find the weak closure of $S$.

**42.** For $1 \le m < n$ let $x_{mn}$ be the sequence

$$x_{mn}(j) = \begin{cases} 1, & \text{if } j = m \\ m, & \text{if } j = n \\ 0, & \text{otherwise.} \end{cases}$$

Let $E = \big\{x_{mn} \;\big|\; 1 \le m < n\big\} \subseteq \ell^2$. Prove that 0 is in the weak closure of $E$, but that no sequence in $E$ converges weakly to 0.

**43.** Let $\ell^\infty$ be the Banach space of all bounded complex sequences, with the supremum norm. For each $n \ge 1$ define $\phi_n \in (\ell^\infty)^*$ by

$$\phi_n(x) = \frac{1}{n} \sum_{j=1}^{n} x_j.$$

Let $E = \{\phi_1,\ \phi_2,\ \ldots\} \subseteq (\ell^\infty)^*$. Prove that

(i) $E$ has a weak$^*$ cluster point in $(\ell^\infty)^*$.
(ii) No weak$^*$ cluster point of $E$ lies in $\kappa(\ell^1)$.
(iii) Let $T : \ell^\infty \to \ell^\infty$ be the shift map: $(Tx)_j = x_{j+1}$. If $\phi$ is a weak$^*$ cluster point of $E$, then $\phi \circ T = \phi$.