MAT 473 **Spring, 2005**

NOTES ON THE IMPLICIT FUNCTION THEOREM

The implicit function theorem gives sufficient conditions for a level set of a function to define some of the variables as functions of the others near a point on the level set, and for this "implicitly defined" function to be differentiable. In the case of a function from $\mathbf{R}^2$ to $\mathbf{R}$, where the level set is curve in the plane, the theorem is easy to visualize, and a direct proof is not too hard to construct. This proof can be generalized to higher dimensions. We will give a different proof using the contraction mapping theorem. In what follows we will identify $\mathbf{R}^{n+m}$ with the Cartesian product $\mathbf{R}^n \times \mathbf{R}^m$. Thus $(x, y) \in \mathbf{R}^{n+m}$ means that $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. We will use the notation D_1 for the derivative with respect to the first n variables, and D_2 for the derivative with respect to the last m variables. Thus if f is a function defined on $\mathbf{R}^{n+m}$ we will write $f'(a) = \big(D_1 f(a),\, D_2 f(a)\big)$.

Theorem. (Implicit function theorem.) Let $f : U \subseteq \mathbf{R}^{n+m} \to \mathbf{R}^m$ be a C^1 function, where U is open, and let $(x_0, y_0) \in U$. Let $A = D_1 f(x_0, y_0) \in M_{m\times n}$, $B = D_2 f(x_0, y_0) \in M_m$, and $c = f(x_0, y_0) \in \mathbf{R}^m$. Suppose that B is invertible. Then there are r, $s > 0$ such that $B_r(x_0) \times B_s(y_0) \subseteq U$, and

(i) for each $x \in B_r(x_0)$ there is a unique $y \in B_s(y_0)$ such that $f(x, y) = c$;

(ii) Define $g : B_r(x_0) \to B_s(y_0)$ by $f\big(x, g(x)\big) = c$ (possible by part (i)). Then g is a C^1 function.

For the proof we need a C^1 version of the characterization of differentiability. We present this in general before proving the theorem. Let $h : U \subseteq \mathbf{R}^k \to \mathbf{R}$ be C^1, with U open. Define $\phi : U \times U \to \mathbf{R}$ by

$$\phi(x, y) = \begin{cases} \frac{h(y)-h(x)-h'(x)(y-x)}{\|y-x\|}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

We claim that ϕ is continuous on $U \times U$. To see this we apply the mean value theorem to the expression $h(y) - h(x)$ to get a number $\theta \in (0, 1)$ such that

$$h(y) - h(x) = h'\big(x + \theta(y - x)\big)(y - x).$$

Then we find for x, $y \in U$ with $x \neq y$,

$$\phi(x, y) = \frac{h'\big(x + \theta(y - x)\big)(y - x) - h'(x)(y - x)}{\|y - x\|} = \Big[\big(h'(x + \theta(y - x)\big) - h'(x)\Big] \frac{(y - x)}{\|y - x\|}.$$

Therefore $\big|\phi(x, y)\big| \leq \Big|\big(h'(x + \theta(y - x)\big) - h'(x)\Big|$. In fact, this inequality is true for all x, $y \in U$, even if $x = y$. Since h' is continuous, we get $\lim_{(x,y)\to(x_0,x_0)} \phi(x, y) = 0$, and thus ϕ

is continuous at (x_0, x_0). Since ϕ is clearly continuous at points (x_0, y_0) for which $x_0 \neq y_0$, ϕ is continuous on all of $U \times U$. Finally, from the definition of ϕ we have

$$f(y) = f(x) + f'(x)(y - x) + \phi(x, y)\|y - x\|$$

for all x, $y \in U$.

Proof of theorem. Define ϕ on $U \times U$ as above for the function f. (Note that ϕ takes values in $\mathbf{R}^m$ instead of $\mathbf{R}$. The above discussion implies that each component function of ϕ is continuous, and hence so is ϕ.) Choose $s > 0$ such that

(1) $\overline{B}_s(x_0) \times \overline{B}_s(y_0) \subseteq U$,

(2) $\big\|B - D_2 f(x, y)\big\| < \dfrac{1}{4\|B^{-1}\|}, \quad (x, y) \in \overline{B}_s(x_0) \times \overline{B}_s(y_0)$,

(3) $\big\|\phi\big((x_1, y_1), (x_2, y_2)\big)\big\| < \dfrac{1}{4\|B^{-1}\|}, \quad (x_i, y_i) \in \overline{B}_s(x_0) \times \overline{B}_s(y_0)$.

Choose $r > 0$, $r \leq s$, such that

(4) $\big\|f(x, y_0) - f(x_0, y_0)\big\| < \dfrac{s}{2\|B^{-1}\|}, \quad x \in B_r(x_0)$.

We will imitate Newton's method to solve the equation $f(x, y) = c$ for y, given a fixed x. For $x \in B_r(x_0)$ define $K_x : \overline{B}_s(y_0) \to \mathbf{R}^m$ by

$$K_x(y) = y - B^{-1}\big(f(x, y) - f(x_0, y_0)\big).$$

We will show that K_x is a contraction mapping of $\overline{B}_s(y_0)$. For this we have to prove two things: first, that K_x maps the ball into itself, and second, that it has a contraction constant. We will prove the second item first. Let y, $z \in \overline{B}_s(y_0)$. Then

$$\begin{aligned} K_x(y) - K_x(z) &= (y - z) - B^{-1}\big(f(x, y) - f(x, z)\big) \\ &= (y - z) - B^{-1}\big[D_2 f(x, z)(y - z) + \phi\big((x, y), (x, z)\big)\|y - z\|\big], \end{aligned}$$

by the mean value theorem,

$$= \big(I - B^{-1} D_2 f(x, z)\big)(y - z) - B^{-1}\phi\big((x, y), (x, z)\big)\|y - z\|.$$

Hence

$$\begin{aligned} \|K_x(y) - K_x(z)\| &\leq \Big(\big\|I - B^{-1} D_2 f(x, z)\big\| + \big\|B^{-1}\phi\big((x, y), (x, z)\big)\big\|\Big)\|y - z\| \\ &\leq \|B^{-1}\|\Big(\big\|B - D_2 f(x, z)\big\| + \big\|\phi\big((x, y), (x, z)\big)\big\|\Big)\|y - z\| \\ &\leq \|B^{-1}\|\Big(\frac{1}{4\|B^{-1}\|} + \frac{1}{4\|B^{-1}\|}\Big)\|y - z\| \\ &= \frac{1}{2}\|y - z\|. \end{aligned}$$

Now

$$\begin{aligned}
\|K_x(y)-y_0\| &\le \|K_x(y)-K_x(y_0)\| + \|K_x(y_0)-y_0\| \\
&\le \frac{1}{2}\|y-y_0\| + \big\|B^{-1}\big(f(x,y_0)-f(x_0,y_0)\big)\big\| \\
&\le \frac{1}{2}s + \|B^{-1}\|\frac{s}{2\|B^{-1}\|} \\
&= s.
\end{aligned}$$

By the contraction mapping theorem, K_x has a unique fixed point in $\overline{B}_s(y_0)$. By the definition of K_x, y is a fixed point for K_x if and only if $f(x,y)=c$. This proves (i).

For (ii) we need another estimate. Let $h \in B_r(0)$. Put

$$\begin{aligned}
x &= x_0 + h \\
y &= g(x) = g(x_0+h) \\
\Delta y &= y - y_0 = g(x_0+h) - g(x_0).
\end{aligned}$$

By the definition of g,

$$\begin{aligned}
y &= K_x(y) \\
&= y - B^{-1}\big(f(x,y)-f(x_0,y_0)\big) \\
&= y - B^{-1}\big[A(x-x_0) + B(y-y_0) + \phi\big((x,y),(x_0,y_0)\big)\|(x-x_0,y-y_0)\| \\
&= y_0 - B^{-1}Ah - B^{-1}\phi\big((x,y),(x_0,y_0)\big)\|(h,\Delta y)\|.
\end{aligned}$$

$$\begin{aligned}
\|\Delta y\| &\le \|B^{-1}A\|\cdot\|h\| + \|B^{-1}\|\frac{1}{4\|B^{-1}\|}\big(\|h\|+\|\Delta y\|\big) \\
\frac{3}{4}\|\Delta y\| &\le \big(\|B^{-1}A\| + \frac{1}{4}\big)\|h\| \\
\|\Delta y\| &\le \frac{4}{3}\big(\|B^{-1}A\| + \frac{1}{4}\big)\|h\|.
\end{aligned}$$

Therefore g is continuous at x_0. Moreover we now see that

$$\begin{aligned}
g(x_0+h) - g(x_0) - B^{-1}Ah &= -B^{-1}\cdot\phi\big((x,y),(x_0,y_0)\big)\cdot\|(h,\Delta y)\|, \\
\frac{1}{\|h\|}\big\|g(x_0+h)-g(x_0)-B^{-1}Ah\big\| &\le \frac{1}{\|h\|}\|B^{-1}\|\cdot\big\|\phi\big((x,y),(x_0,y_0)\big)\big\|\cdot\big(\|h\|+\|\Delta y\|\big) \\
&\le \|B^{-1}\|\cdot\|\phi\big((x,y),(x_0,y_0)\big)\|\cdot\big(1+\frac{4}{3}(\|B^{-1}A\|+\frac{1}{4})\big) \\
&\to 0
\end{aligned}$$

as $h \to 0$, since ϕ is continuous and vanishes on the diagonal of $U \times U$. Therefore g is differentiable at x_0, and

$$g'(x_0) = -B^{-1}A = -D_2 f(x_0,y_0)^{-1} D_1 f(x_0,y_0).$$

Now notice that by the inequality defining s, $D_2 f(x, y)$ is invertible for all $(x, y) \in B_r(x_0) \times B_s(y_0)$. Therefore the proof given applies to each point $\big(x, g(x)\big)$. It follows that g is differentiable on $B_r(x_0)$, and its derivative is given by $g'(x) = -D_2 f\big(x, g(x)\big)^{-1} D_1 f\big(x, g(x)\big)$. Since g is continuous on $B_r(x_0)$, we now see that g' is continuous also. Thus g is C^1. ∎

Corollary 1. Let $V = B_r(x_0) \times B_s(y_0)$ and $M = \big\{(x, y) \in \mathbf{R}^{n+m} : f(x, y) = c\big\}$. Then $V \cap M$ is the graph of (the C^1 function) g.

Corollary 2. $g'(x) = -D_2 f\big(x, g(x)\big)^{-1} D_1 f\big(x, g(x)\big)$. (This was derived in the course of the proof, but could be deduced from the equation $f\big(x, g(x)\big) = c$ by the chain rule.)

Corollary 3. If f is C^k then g is also C^k.

Proof. If f is C^2 then $D_i f$ is C^1. Since g is known to be C^1, the expression for g' in Corollary 2 shows that g' is C^1, and hence that g is C^2. This argument may be continued as many times as f is continuously differentiable. ∎

We next sketch how the implicit function theorem can be used to justify the method of Lagrange multipliers for constrained optimization problems. Let $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ be a C^1 function, with $n > m$. Let $S = \{x \in U : f(x) = c\}$, the level set of f for the value $c \in \mathbf{R}^m$. Suppose that $f'(x)$ has rank m at each point of S. Now let $h : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function. Suppose that $h|_S$ has a local extremum at a point $a \in S$. How can we find the point a? Let us write $\mathbf{R}^n = \mathbf{R}^{n-m} \times \mathbf{R}^m$ so that, subject to permuting the variables (which we explain more carefully below, in the proof of the submersion theorem), $D_2 f(a)$ is invertible. The implicit function theorem tells us that near a, S defines x_2 as a function ϕ of x_1: $f\big(x_1, \phi(x_1)\big) = c$ for all x_1 near a_1. Moreover, $g\big(x_1, \phi(x_1)\big)$ has a local extremum at a_1. Thus both functions have derivative 0 at a_1. Differentiating gives

$$f'(a) \begin{pmatrix} I_{n-m} \\ \phi'(a) \end{pmatrix} = 0,$$

$$g'(a) \begin{pmatrix} I_{n-m} \\ \phi'(a) \end{pmatrix} = 0.$$

Let

$$T = \begin{pmatrix} I_{n-m} \\ \phi'(a) \end{pmatrix}_{n \times (n-m).}$$

The columns of T are a basis for an $(n-m)$-dimensional subspace of $\ker f'(a)$. But since $f'(a)$ has rank m, the kernel of $f'(a)$ is $(n-m)$-dimensional. Thus the column space of T equals the kernel of $f'(a)$. But $g'(a)$ is orthogonal to the columns of T, and hence is orthogonal to the kernel of $f'(a)$. This means that $g'(a)$ lies in the row space of $f'(a)$. Therefore there are scalars $\lambda_1, \ldots, \lambda_m$ such that $g'(a) = \lambda_1 f_1'(a) + \cdots + \lambda_m f_m'(a)$. We have that

$$D_j g(a) = \sum_{i=1}^{m} \lambda_i D_j f_i(a), \quad 1 \leq j \leq n$$

$$f_k(a) = c_k, \quad 1 \leq k \leq m$$

form a system of $m+n$ equations in the $m+n$ variables $a_1, \ldots, a_n, \lambda_1, \ldots, \lambda_m$.

We now use the implicit function theorem to prove the inverse function theorem.

Theorem. (Inverse function theorem.) Let $U \subseteq \mathbf{R}^n$ be open, let $a \in U$, and let $f : U \to \mathbf{R}^n$ be C^1. Suppose that $f'(a)$ is invertible. Then there are open sets V and $W \subseteq \mathbf{R}^n$, with $a \in V \subseteq U$ and $f(a) \in W$, such that $f : V \to W$ is one-to-one and onto with C^1 inverse.

Proof. Define $h : \mathbf{R}^n \times U \to \mathbf{R}^n$ by $h(x,y) = f(y) - x + f(a)$. Then $h\big(f(a), a\big) = f(a)$, and $D_2 h\big(f(a), a\big) = f'(a)$ is invertible. The implicit function theorem implies that there are r, $s > 0$ such that $B_r\big(f(a)\big) \times B_s(a) \subseteq \mathbf{R}^n \times U$, and such that:

(i) for all $x \in B_r\big(f(a)\big)$ there exists a unique $y \in B_s(a)$ such that $h(x,y) = f(a)$,

(ii) define $g : B_r\big(f(a)\big) \to B_s(a)$ by $h\big(x, g(x)\big) = f(a)$. Then g is C^1.

Now the definition of g means that $f\big(g(x)\big) - x + f(a) = f(a)$, or

$$f\big(g(x)\big) = x, \quad x \in B_r\big(f(a)\big).$$

Since f and g are C^1 we find that $f'\big(g(x)\big) \cdot g'(x) = I_n$ for $x \in B_r\big(f(a)\big)$. Thus $g'(x)$ is invertible for all $x \in B_r\big(f(a)\big)$. Notice also that

$$f(U) \supseteq f\big(B_s(a)\big) \supseteq f\Big(g\big(B_r(f(a))\big)\Big) = B_r\big(f(a)\big).$$

Thus the hypotheses imply that $f(U)$ is a neighborhood of $f(a)$ whenever $f'(a)$ is invertible. Applying this fact to g we find that $g\big(B_r(f(a))\big)$ is a neighborhood of $g(x)$ for all $x \in B_r\big(f(a)\big)$; i.e. $V = g\big(B_r(f(a))\big)$ is an open set. Let $W = B_r\big(f(a)\big)$. Then $g : W \to V$ is onto, and is one-to-one by the equation $f \circ g = id$. Therefore f is one-to-one and onto. ∎

Definition. A C^1 map that is one-to-one and onto with C^1 inverse is called a *diffeomorphism.*

Thus if $U \subseteq \mathbf{R}^n$ is open, $f : U \to \mathbf{R}^n$ is C^1, and if $f'(x)$ is invertible for all x, then f is locally a diffeomorphism.

Corollary. Let $U \subseteq \mathbf{R}^n$ be open, $f : U \to \mathbf{R}^n$ a C^1 map, and suppose that $f'(x)$ is invertible for all x in U. Then f is an open mapping. (That is, $f(E)$ is an open set whenever $E \subseteq U$ is an open set.)

We think of a diffeomorphism as giving a smooth change of coordinates. There is an important branch of mathematics, *differential topology*, studying properties that are not altered by diffeomorphisms. *Manifolds* are defined by such properties.

Example. Let $k \geq \ell$. The function $\pi : \mathbf{R}^k \to \mathbf{R}^\ell$ defined by $\pi(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$ is called *the canonical submersion.* Thinking of $\mathbf{R}^k$ as $\mathbf{R}^\ell \times \mathbf{R}^{k-\ell}$, we have $\pi'(x) = \big(I_\ell, 0_{k-\ell}\big)$. Thus $D_1\pi$ is invertible. Thus π is the simplest map to which the implicit function theorem applies. For $a \in \mathbf{R}^k$ let $b = \pi(a) = (a_1, \ldots, a_\ell) \in \mathbf{R}^\ell$. The level set $\pi^{-1}(b) = \big\{\pi(a)\big\} \times \mathbf{R}^{k-\ell}$ is the graph of the constant function $g : \mathbf{R}^{k-\ell} \to \mathbf{R}^\ell$ given by $g(y) = b$, in the sense that the graph of g may be viewed as

$$\big\{\big(g(y), y\big) \in \mathbf{R}^\ell \times \mathbf{R}^{k-\ell} : y \in \mathbf{R}^{k-\ell}\big\} = \big\{(b, y) : y \in \mathbf{R}^{k-\ell}\big\}.$$

The next theorem shows that every *submersion* is locally the canonical submersion, up to diffeomorphism.

Definition. A C^1 map $f : U \subseteq \mathbf{R}^k \to \mathbf{R}^\ell$, U open, is a *submersion at* $a \in U$ if the rank of $f'(a)$ equals ℓ. f is called a *submersion* if it is a submersion at each $x \in U$.

Submersion Theorem. Let $k > \ell$, $U \subseteq \mathbf{R}^k$ open, $a \in U$, and $f : U \to \mathbf{R}^\ell$ a C^1 map such that the rank of $f'(a)$ equals ℓ. Then there are open sets V, $W \subseteq \mathbf{R}^k$ with $a \in W \subseteq U$, and a diffeomorphism $h : V \to W$, such that $f \circ h = \pi|_V$.

Proof. We will permute the coordinates so that we can "see" where the rank condition holds. Since $f'(a)$ has rank ℓ, it has ℓ linearly independent columns, say i_1, ..., i_ℓ. In other words, the $\ell \times \ell$ matrix $A = \big(D_{i_1} f(a), \ldots, D_{i_\ell} f(a)\big)$ is invertible. Let $\{j_1, \ldots, j_{k-\ell}\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_\ell\}$. Let $B = \big(D_{j_1} f(a), \ldots, D_{j_{k-\ell}} f(a)\big) \in M_{\ell\times(k-\ell)}$. Put $T = \big(e_{i_1}, \ldots, e_{i_\ell}, e_{j_1}, \ldots, e_{j_{k-\ell}}\big) \in M_k$, a permutation matrix. Then $f'(a)\cdot T = \big(A\ B\big)$. Consider the function $f \circ T : T^{-1}(U) \to \mathbf{R}^\ell$. Note that

$$(f \circ T)'\big(T^{-1}(a)\big) = f'\big(T(T^{-1}(a))\big) \cdot T'\big(T^{-1}(a)\big) = f'(a) \cdot T.$$

We will use the inverse function theorem. Define $g : T^{-1}(U) \to \mathbf{R}^k$ by

$$g(x) = \big(f \circ T(x),\ x_{\ell+1}, \ldots, x_k\big).$$

Thus $\pi \circ g = f \circ T$. Note also that $g'(T^{-1}(a)) = \begin{pmatrix} A & B \\ 0 & I_{k-\ell} \end{pmatrix}$, an invertible matrix.

Thus there are open sets $Z_1, Z_2 \subseteq \mathbf{R}^k$, with $T^{-1}(a) \in Z_1 \subseteq T^{-1}(U)$, such that $g : Z_1 \to Z_2$ is a diffeomorphism. Define $V = Z_2$, $W = T(Z_1)$, and $h = T \circ g^{-1} : V \to W$. Then since $f \circ T|_{Z_1} = \pi \circ g|_{Z_1}$, we have $f \circ T \circ g^{-1}|_{g(Z_1)} = \pi|_{g(Z_1)}$, or $f \circ h = \pi|_V$. ∎

Definition. A *k-dimensional (C^1) manifold* (in $\mathbf{R}^n$) is a subset $M \subseteq \mathbf{R}^n$ such that for all $a \in M$ there is an open set $W \subseteq \mathbf{R}^n$ with $a \in W$, and a C^1 diffeomorphism $\phi : W \to V$, V an open subset of $\mathbf{R}^n$, such that $\phi(W \cap M) = \phi(W) \cap (\mathbf{R}^k \times 0^{n-k})$ (i.e. $x \in W \cap M$ if and only if $\phi(x) = (x_1, \ldots, x_k, 0, \ldots, 0)$.)

Example. Let $f : \mathbf{R}^n \to \mathbf{R}^{n-k}$ be C^1, let $c \in \mathbf{R}^{n-k}$, and set $M = f^{-1}(c)$. Suppose that f is a submersion at each point of M. Then M is a k-dimensional manifold. To see this, replace $f(x)$ by $f(x) - c$, so that we may assume $c = 0$. For $a \in M$ there are open V, $a \in W \subseteq \mathbf{R}^n$ and a diffeomorphism $h : V \to W$ such that $f \circ h = \pi$ on V. Put $\phi = h^{-1}$. Then

$$\begin{aligned}
\phi(W \cap M) &= h^{-1}(W \cap M) \\
&= V \cap h^{-1}(M) \\
&= V \cap h^{-1}(f^{-1}(0)) \\
&= V \cap (f \circ h)^{-1}(0) \\
&= V \cap \pi^{-1}(0) \\
&= V \cap (\mathbf{R}^k \times \{0\}).
\end{aligned}$$

Examples of manifolds.

1. Define $f : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f(x) = \|x\|^2$. It is easy to check that $f'(x)$ is onto for all $x \neq 0$. Thus if we define $S^n = f^{-1}(1)$ we find that the *n-sphere* is an n-dimensional manifold (in $\mathbf{R}^{n+1}$).

2. Consider the determinant function $\det : M_n \to \mathbf{R}$. We claim that if $A \in M_n$ is invertible then $\det'(a)$ is onto (exercise). Therefore $SL(n, \mathbf{R}) = \det^{-1}(1)$ is an $(n^2 - 1)$-dimensional manifold.

3. Let S_n denote the set of $n \times n$ symmetric matrices. It is easy to see that S_n is just $\mathbf{R}^{n(n+1)/2}$ (in the same way that M_n is just $\mathbf{R}^{n^2}$). Define $f : M_n \to S_n$ by $f(a) = A^t A$. The *orthogonal group* is given by $O(n) = f^{-1}(I_n)$. It is an exercise to check that f is a submersion at each orthogonal matrix, and therefore $O(n)$ is a manifold (of dimension $n(n-1)/2$).