MAT 473 **HOMEWORK 5** **Due: 2/25/05**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions.

17. Let S be the set of all rank-one 2×2 matrices, a subset of $M_2 \cong \mathbf{R}^4$. Prove that S is a 3-dimensional manifold. (Hint: Let $U \subseteq M_2$ be the open set of all non-zero 2×2 matrices. Let $\det : U \to \mathbf{R}$ be the determinant function restricted to U. Prove that det is a submersion on U.)

18. Let $U \subseteq \mathbf{R}^n$ be an open set, and let $P \subseteq U$ be a special polygon. Prove that there is a special polygon Q such that $P \subseteq \operatorname{int}(Q)$ and $Q \subseteq U$. (Hint: if A and B are disjoint closed nonempty sets with A compact, then $\operatorname{dist}(A, B) > 0$.)

19. Let $U \subseteq \mathbf{R}^n$ be a nonempty open set. Prove that there exist non-overlapping non-degenerate special rectangles $E_1, E_2, \ldots$ such that $U = \cup_{j=1}^{\infty} E_j$. (Hint: Let $\mathcal{C}_0$ be the collection of closed cubes in $\mathbf{R}^n$ having edge 1 and each vertex in $\mathbf{Z}^n$. Select those cubes of $\mathcal{C}_0$ that are contained in U. Bisect the edges of the remaining cubes to get a new collection of cubes of edge 1/2. Select among these cubes the ones contained in U. Continue inductively.)

20. (i) Let $E \subseteq U \subseteq \mathbf{R}^n$, where E is a nondegenerate special rectangle and U is an open set, and let $\epsilon > 0$ be given. Prove that there is a nondegenerate special rectangle $\widetilde{E}$ such that $E \subseteq \operatorname{int}(\widetilde{E})$, $\widetilde{E} \subseteq U$, and $\lambda(\widetilde{E}) - \lambda(E) < \epsilon$.

(ii) Let $E_1, E_2, \ldots$ be non-overlapping nondegenerate special rectangles contained in an open set U such that $U = \cup_{i=1}^{\infty} E_i$. Prove that $\lambda(U) = \sum_{i=1}^{\infty} \lambda(E_i)$. (Hint: to prove $\leq$, let $P \subseteq U$ be a special polygon. Use part (i) to choose $\widetilde{E_i}$ for $\epsilon/2^i$. Then use compactness of P.)