Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions.

9. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be C^2. Prove that

$$D_1 D_1 f(0,0) = \lim_{h \to 0^+} \frac{f(2h, e^{-1/2h}) - 2f(h, e^{-1/h}) + f(0,0)}{h^2}.$$

(Hint: $\lim_{h\to 0^+} e^{-1/h}/h^2 = \lim_{t\to\infty} t^2 e^{-t}$.)

10. (i) Let (X, d) be a metric space, let $T : X \to M_n$ be a continuous function, and let $x_0 \in X$. Suppose that $T(x_0)$ has a positive eigenvalue and a negative eigenvalue. Prove that there are unit vectors v_+, $v_- \in \mathbf{R}^n$ such that

$$\begin{aligned} \langle T(x)v_+, v_+\rangle &> 0 \\ \langle T(x)v_-, v_-\rangle &< 0 \end{aligned}$$

for all x near x_0.

(ii) Let $U \subseteq \mathbf{R}^n$ be open, $a \in U$, let $f : U \to \mathbf{R}$ be a C^2 function, and suppose $f'(a) = 0$. Suppose further that $f''(a)$ is neither positive nor negative semidefinite. Prove that f does not have a local extremum at a.

11. Prove that there exists a nonempty complete metric space (X, d), and a function $f : X \to X$, such that $d\big(f(x), f(y)\big) < d(x, y)$ whenever $x \neq y$, and such that f has no fixed point in X.

12. Prove that $\sup\left\{ \dfrac{\|A\|_2}{\|A\|} : A \in M_{m\times n},\ A \neq 0 \right\} = \min\{\sqrt{m}, \sqrt{n}\}$.

(Suggestions: Recall that a matrix U is an *isometry* if $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all x and y. (Notice that since isometries preserve length of vectors they must be one-to-one maps; hence $M_{m\times n}$ contains isometries if and only if $n \le m$.)

(i) Prove that if U is an isometry then for any matrix A for which UA is defined, $\|UA\| = \|A\|$ and $\|UA\|_2 = \|A\|_2$. (For the computation with $\|\cdot\|_2$, write $A = (A_{:1}, \dots, A_{:n})$ in terms of its columns.)

(ii) Use (part of) the *polar decomposition* from linear algebra: if $A \in M_{m\times n}$ with $n \le m$ then there are an isometry $U \in M_{m\times n}$ and a positive semidefinite $T \in M_n$ such that $A = UT$.)