MAT 473 **HOMEWORK 2** **Due: 2/4/05**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions.

5. Let $U \subseteq \mathbf{R}^n$ be open, let $f : U \to \mathbf{R}^n$, and let $a \in U$. Suppose that f is differentiable at a, and that $f'(a)$ is a non-singular linear transformation. Prove that there is a number $r > 0$ such that for all $x \in U$, if $0 < \|x - a\| < r$ then $f(x) \neq f(a)$.

6. Let $f_1, \ldots, f_n : \mathbf{R}^n \to \mathbf{R}$ be continuously differentiable functions, and suppose that $D_i f_j(x) = D_j f_i(x)$ for all i and j, and for all $x \in \mathbf{R}^n$. Prove that there exists a function $F : \mathbf{R}^n \to \mathbf{R}$ such that $f_i = D_i F$ for all i. (Hint: let $a \in \mathbf{R}^n$ be a fixed point. Define F by

$$F(x) = \int_{a_1}^{x_1} f_1(t, a_2, \ldots, a_n)\, dt + \int_{a_2}^{x_2} f_2(x_1, t, a_3, \ldots, a_n)\, dt + \cdots + \int_{a_n}^{x_n} f_n(x_1, \ldots, x_{n-1}, t)\, dt.)$$

7. Let $f_1, f_2 : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}$ be given by

$$f_1(x) = \frac{-x_2}{x_1^2 + x_2^2}, \qquad f_2(x) = \frac{x_1}{x_1^2 + x_2^2}.$$

(i) Prove that $D_1 f_2 = D_2 f_1$ on $\mathbf{R}^2 \setminus \{0\}$.

(ii) Prove that there does not exist a continuously differentiable function $F : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}$ such that $f_i = D_i F$ for $i = 1, 2$. (Hint: Let $g : [0, 2\pi] \to \mathbf{R}^2 \setminus \{0\}$ be given by $g(t) = (\cos t, \sin t)$. Apply the mean value theorem to $F(g(t))$.)

8. Prove that the following function $f : \mathbf{R}^2 \to \mathbf{R}$ is continuously differentiable, and that all second-order partial derivatives of f exist at the origin, but that $D_1 D_2 f(0) \neq D_2 D_1 f(0)$:

$$f(x) = \begin{cases} \dfrac{x_1^3 x_2}{x_1^2 + x_2^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$