MAT 473 **HOMEWORK 12** **Due: 4/29/05**

Write neatly, not too small, and not too lightly. You may discuss the problems with other students from class, but you must write your own solutions.

45. Let $f,\, g \in L^1(\mathbf{R}^n)$. Define $h : \mathbf{R}^{2n} \to \mathbf{R}$ by $h(x,y) = f(x-y)g(y)$.

(i) Prove that h is measurable. (Hint: consider $h \circ T$, where $T : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is the linear map given by $T(x,y) = (x+y, y)$.)

(ii) Prove that h is integrable. (Hint: use T to change variables.)

(iii) Define $f * g : \mathbf{R}^n \to \mathbf{R}$ by

$$f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy.$$

Prove that $f * g$ is integrable, and that $\|f * g\|_1 \le \|f\|_1\,\|g\|_1$.

46. Let $h \in L^1(\mathbf{R}^n)$, and let $a = \int h$. For $t > 0$ define $h^t(y) = t^n h(ty)$.

(i) Prove that h^t is integrable, and $\int h^t = a$.

(ii) Let $f \in L^1(\mathbf{R}^n)$. Prove that

$$\lim_{t \to \infty} \|f * h^t - af\|_1 = 0.$$

(Hints: choose r such that $\int_{\{\|y\|>r\}} \big|h(y)\big|\,dy$ is small. Considering $\|f * h^t - af\|_1$ estimate separately the integrals over $B_{r/t}(0)$ and $B_{r/t}(0)^c$. For the first of these, use the continuity of translation in L^1.)

47. Let f be an integrable function on $\mathbf{R}$.

(i) Prove that for each $t > 0$ the following integral exists:

$$g(t) = \int_{\mathbf{R}} e^{-t/x^2} f(x)\,dx.$$

(ii) Prove that the function $g : (0,\infty) \to \mathbf{R}$ defined in part (i) is continuous.

(iii) Find $\lim_{t\to\infty} g(t)$ and $\lim_{t\to 0^+} g(t)$, and justify your answers.