MAT 372 **HOMEWORK 5** **Due: 3/4/05**

Please write **neatly** and **legibly**, write on **only one side of the paper**, print your name, and STAPLE the pages together before coming to class. Always show your work.

21. Let $E \subseteq \mathbf{R}^n$ be an open set, and let $f : E \to \mathbf{R}$. Suppose that $D_1 f$, ..., $D_n f$ exist and are bounded in E. Prove that f is continuous in E. (Hint: imitate the proof of Theorem 2.5.)

22. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}$ be twice continuously differentiable. Prove the following:

$$D_i(f \circ T) = \sum_j (D_i T_j)\big((D_j f) \circ T\big).$$

$$D_{ii}(f \circ T) = \sum_j (D_{ii} T_j)\big((D_j f) \circ T\big) + \sum_{j,k} (D_i T_j)(D_i T_k)\big((D_{jk} f) \circ T\big).$$

$$\Delta(f \circ T) = \sum_j (\Delta T_j)\big((D_j f) \circ T\big) + \sum_{j,k} (\nabla T_j \cdot \nabla T_k)\big((D_{jk} f) \circ T\big).$$

23. Define the change from Cartesian to polar coordinates by

$$r(x, y) = \sqrt{x^2 + y^2}$$

$$\theta(x, y) = \tan^{-1}\big(\frac{y}{x}\big)$$

$$T(x, y) = \big(r(x, y), \theta(x, y)\big).$$

Compute ∇r, $\nabla \theta$, Δr, $\Delta \theta$, and $\Delta(f \circ T)$ for a twice continuously differentiable function $f(r, \theta)$. (Hint: Problem 22 should be your outline.) If you are ambitious, you can work out in an analogous manner the formula for the Laplacian in spherical coordinates. The answer is printed in an exercise in the text.

24. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(\mathbf{x}) = \begin{cases} \dfrac{x_1 x_2 (x_1^2 - x_2^2)}{x_1^2 + x_2^2}, & \text{if } \mathbf{x} \neq 0 \\ \mathbf{0}, & \text{if } \mathbf{x} = 0. \end{cases}$$

Prove the following:

(i) f, $D_1 f$, $D_2 f$ are continuous in $\mathbf{R}^2$.
(ii) $D_{12} f$ and $D_{21} f$ exist in all of $\mathbf{R}^2$.
(iii) $D_{12} f(\mathbf{0}) \neq D_{21} f(\mathbf{0})$.

(OVER)

25. Let $0 < r < R$ and define $f : \mathbf{R}^2 \to \mathbf{R}^3$ by

$$f(\theta, \alpha) = \Big((R + r\cos\alpha)\cos\theta,\ (R + r\cos\alpha)\sin\theta,\ r\sin\alpha\Big).$$

(The range, T, of f is a *torus*.)

(i) Find all points of the form $f(\theta, \alpha) \in T$ such that $\nabla f_1(\theta, \alpha) = \mathbf{0}$. (Hint: your answer will be a finite subset of $\mathbf{R}^3$.)

(ii) Show that one of the points in part (i) correponds to a local maximum of f_1, one corresponds to a local minimum of f_1, and the others are neither local maxima nor minima.

(iii) Find all points of the form $f(\theta, \alpha) \in T$ such that $\nabla f_3(\theta, \alpha) = \mathbf{0}$.

(iv) Which points in part (iii) correspond to local maxima and which to local minima? Explain your answer.