

Please write **neatly** and **legibly**, write on **only one side of the paper**, print your name, and STAPLE the pages together before coming to class. Always show your work.

**16.** Let  $f : \mathbf{R} \rightarrow \mathbf{R}^n$  be a differentiable curve, with  $f'(t) \neq 0$  for all  $t \in \mathbf{R}$ . Let  $y_0 \in \mathbf{R}^n$  be a fixed point not on the curve (i.e.  $f(t) \neq y_0$  for all  $t \in \mathbf{R}$ ). Suppose that  $t_0 \in \mathbf{R}$  is such that  $f(t_0)$  is closer (or equally close) to  $y_0$  than any other point on the curve. Prove that the vector  $f(t_0) - y_0$  is orthogonal to the velocity vector  $f'(t_0)$ . (Hint: consider the function  $\phi : \mathbf{R} \rightarrow \mathbf{R}$  defined by  $\phi(t) = |f(t) - y_0|^2$ .)

**17.** Consider a particle which moves on a circular helix in  $\mathbf{R}^3$  with position vector

$$f(t) = (a \cos \omega t, a \sin \omega t, b\omega t).$$

- (i) Show that the speed of the particle is constant. (The *speed* of a particle with position vector  $f(t)$  is  $|f'(t)|$ .)
- (ii) Show that its velocity vector makes a constant nonzero angle with the  $z$ -axis.
- (iii) Prove that there does not exist a number  $c$  with  $0 < c < 2\pi/\omega$  such that

$$f\left(\frac{2\pi}{\omega}\right) - f(0) = f'(c)\left(\frac{2\pi}{\omega} - 0\right).$$

(Thus the obvious generalization of the mean value theorem to higher dimensions is not valid.)

**18.** Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be given by

$$f(x) = \begin{cases} \frac{x_1^2 x_2}{|x|^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that all directional derivatives of  $f$  exist at 0, and that  $D_v f(0)$  is not a linear function of  $v$ .

**19.** Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be given by

$$f(x) = \begin{cases} \frac{x_1^3 x_2}{x_1^4 + x_2^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that all directional derivatives of  $f$  exist at 0, that  $D_v f(0)$  is a linear function of  $v$ , and that  $f$  is not differentiable at 0. (Hint: to show that  $f$  is not differentiable at 0, assume for the sake of a contradiction that it is. Then you can compute the derivative  $f'(0)$  using partial derivatives. Show that the limit required by the definition of the derivative does not exist by considering points of the form  $(t, t^2)$  with  $t \rightarrow 0$ .)

**20.** Let  $f : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$  be given by the dot product:  $f(x, y) = x \cdot y$  (here we implicitly identify  $\mathbf{R}^n \times \mathbf{R}^n$  with  $\mathbf{R}^{2n}$ ). Fix  $(a, b) \in \mathbf{R}^{2n}$  and define a linear map  $L : \mathbf{R}^{2n} \rightarrow \mathbf{R}$  by  $L(x, y) = a \cdot y + b \cdot x$ . Show that  $df_{(a,b)} = L$  by proving that

$$\lim_{(h,k) \rightarrow 0} \frac{f(a+h, b+k) - f(a, b) - L(h, k)}{|(h, k)|} = 0.$$

(Remember that  $x, y, a, b, h, k$  in the above are all elements of  $\mathbf{R}^n$ .)