MAT 372 **HOMEWORK 2** **Due: 2/4/05**

Please write **neatly** and **legibly**, write on **only one side of the paper**, print your name, and STAPLE the pages together before coming to class. Always show your work.

6. (i) Prove that the open subsets of $\mathbf{R}^n$ satisfy the three definitions of a *topology*:
(ia) $\mathbf{R}^n$ and $\emptyset$ are open,
(ib) If $\{A_i\}_{i\in I}$ is any collection of open subsets of $\mathbf{R}^n$, then $\cup_{i\in I} A_i$ is open,
(ic) If A_1, ..., A_k are open subsets of $\mathbf{R}^n$ then $A_1 \cap \cdots \cap A_k$ is an open set.

(ii) Give an example (with proof) of an infinite collection of open sets whose intersection is not open.

7. Let $a \in \mathbf{R}^n$ and $r > 0$. Prove that $\overline{B_r(a)} = \overline{B}_r(a)$.

8. Let $A \subseteq \mathbf{R}^n$. A point $a \in \mathbf{R}^n$ is a *boundary point* of A if every open ball centered at a intersects both A and A^c. The *boundary* of A, written ∂A, is defined to be the set of all boundary points of A. Let $S_r(a) = \{x \in \mathbf{R}^n : |x - a| = r\}$, the *sphere* with center a and radius r. Prove that $\partial B_r(a) = S_r(a)$.

9. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is not continuous at $(0,0)$. (Hint: consider the restriction of f to various lines through the origin.)

10. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{2x^2y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Let $\phi(t) = (at, bt)$ for constants a and b, and let $\psi(t) = (t, t^2)$.
(i) Show that $\lim_{t\to 0} f\big(\phi(t)\big) = 0$. (Thus f is continuous on every straight line through the origin.)
(ii) Show that $\lim_{t\to 0} f\big(\psi(t)\big) = 1$.
(iii) Explain why f is not continuous at $(0,0)$.