MAT 372 **HOMEWORK 11** **Due: 4/29/05**

Please write **neatly** and **legibly**, write on **only one side of the paper**, print your name, and STAPLE the pages together before coming to class. Always show your work.

In all of the following we assume that the definition of integrability includes being bounded and having bounded support.

43. For each $n = 1, 2, \ldots$, and each $r > 0$, let B_r^n denote the ball in $\mathbf{R}^n$ with radius r and center at the origin. It is immediate that $v(B_r^1) = 2r$.

(i) We all know that $v(B_r^2) = \pi r^2$. Use Cavalieri's principle to prove this.

(ii) Use part (i) and Cavalieri's principle to compute $v(B_r^3)$.

(iii) Use part (ii) and Cavalieri's principle to compute $v(B_r^4)$.

44. (This problem is needed as an ingredient for problem 45.) Let $f : [0, \infty) \to \mathbf{R}$ be a bounded continuous function. Prove that for any $\epsilon > 0$ there is $q_0 > 0$ such that for all $q \geq q_0$ and for all $p > 0$,

$$\left| \int_0^p f(t) e^{-qt}\, dt \right| < \epsilon.$$

(That is to say, $\lim_{q\to\infty} \int_0^p f(t)e^{-qt}\, dt = 0$ *uniformly* in p.)

(Note that it is not the case that e^{-qt} converges uniformly to 0 on $[0, p]$. However you can easily estimate $\left|\int_0^p f(t)e^{-qt}\, dt\right|$ independently of p.)

45. In this problem you will prove that $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$.

(i) Let $a, b > 0$. Apply Fubini's theorem to the function $g(x, y) = \sin x e^{-xy}$ on $[0, a] \times [0, b]$ to prove that

$$\int_0^a \frac{\sin x}{x}\, dx - \int_0^a \frac{\sin x}{x} e^{-bx}\, dx$$
$$= \tan^{-1} b - \cos a \int_0^b \frac{1}{1+y^2} e^{-ay}\, dy - \sin a \int_0^b \frac{y}{1+y^2} e^{-ay}\, dy.$$

(ii) Let $\epsilon > 0$. Prove that there is a_0 such that for all $a \geq a_0$,

$$\left|\int_0^a \frac{\sin x}{x}\, dx - \frac{\pi}{2}\right| < \epsilon.$$

(Hint: you will need to use problem 44 several times.)