MAT 372 **HOMEWORK 10** **Due: 4/22/05**

Please write **neatly** and **legibly**, write on **only one side of the paper**, print your name, and STAPLE the pages together before coming to class. Always show your work.

In all of the following we assume that the definition of integrability includes being bounded and having bounded support.

40. (i) Let f be a non-negative integrable function. Prove that f^2 is integrable. (Hint: use the theorem on approximation by step functions.)

(ii) Let f be an integrable function (not necessarily non-negative). Prove that f^2 is integrable.

(iii) Let f and g be integrable functions. Prove that fg is integrable. (Hint: write fg in terms of $(f \pm g)^2$.)

41. Let $f(x,y)$ be defined on the interval $I = [a,b] \times [c,d]$. Suppose that f is continuous on I, and that $\frac{\partial f}{\partial y}(x,y)$ exists and is continuous on I.

For $y \in [c,d]$ let $G(y) = \int_a^b f(x,y)\,dx$. Prove that G is differentiable, and that $G'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$. Hints:

(1) Let $\epsilon > 0$ be given. Then (prove that) there is $\delta > 0$ such that for all $x \in [a,b]$ and $y_1, y_2 \in [c,d]$, if $|y_1 - y_2| < \delta$ then $\left|\frac{\partial f}{\partial y}(x,y_2) - \frac{\partial f}{\partial y}(x,y_1)\right| < \epsilon$.

(2) Use the mean value theorem to prove that for any $(x,y) \in I$ and $0 < h < \delta$ such that $y + h \in [c,d]$,

$$\left|\frac{f(x,y+h) - f(x,y)}{h} - \frac{\partial f}{\partial y}(x,y)\right| < \epsilon.$$

(3) If $0 < h < \delta$ and $y,\ y+h \in [c,d]$ then

$$\left|\frac{G(y+h) - G(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dy\right| < \epsilon(b-a).$$

42. Let $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ be continuous, and let $g(y)$ and $h(y)$ be differentiable. Find an expression involving f, $\frac{\partial f}{\partial y}$, g, h, g', and h' for:

$$\frac{d}{dy}\int_{g(y)}^{h(y)} f(x,y)\,dx.$$