Riemann Integrability of Continuous and Monotone Functions

In these notes we will use the Cauchy criterion for Riemann integrability (Theorem 5.8) to prove that continuous functions and monotone functions are Riemann integrable. First we derive a simple inequality relating Riemann sums for two tagged partitions, one of which refines the other.

Let ${}^tP$ and ${}^tQ$ be tagged partitions with $P \subseteq Q$. We write

$$\begin{aligned} {}^tP &= \big\{\big(s_i, [x_{i-1}, x_i]\big) : 1 \le i \le m\big\} \\ {}^tQ &= \big\{\big(t_j, [y_{j-1}, y_j]\big) : 1 \le j \le n\big\}. \end{aligned}$$

Because Q refines P, each subinterval $[x_{i-1}, x_i]$ for P is the union of one or more subintervals $[y_{j-1}, y_j]$ for Q. We need some notation for this relationship: for each i between 1 and m, let

$$E_i = \big\{j : [y_{j-1}, y_j] \subseteq [x_{i-1}, x_i]\big\}.$$

It follows that for each i between 1 and m,

$$[x_{i-1}, x_i] = \bigcup_{j \in E_i} [y_{j-1}, y_j].$$

Now we are ready to compare Riemann sums associated with ${}^tP$ and ${}^tQ$. Let $f : [a, b] \to \mathbf{R}$. Then

$$\begin{aligned} S(f, {}^tP) &= \sum_{i=1}^{m} f(s_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{m} f(s_i) \sum_{j \in E_i} (y_j - y_{j-1}) \\ S(f, {}^tQ) &= \sum_{j=1}^{n} f(t_j)(y_j - y_{j-1}) \\ &= \sum_{i=1}^{m} \sum_{j \in E_i} f(t_j)(y_j - y_{j-1}), \end{aligned}$$

and so

$$\begin{aligned} \big|S(f, {}^tP) - S(f, {}^tQ)\big| &= \Big|\sum_{i=1}^{m} \sum_{j \in E_i} \big(f(s_i) - f(t_j)\big)(y_j - y_{j-1})\Big| \\ &\le \sum_{i=1}^{m} \sum_{j \in E_i} \big|f(s_i) - f(t_j)\big|(y_j - y_{j-1}). \end{aligned}$$

We will refer to this inequality as the *preliminary computation*. It is valid whenever ${}^tP$ and ${}^tQ$ are tagged partitions of an interval with $P \subseteq Q$.

Now we will prove the two main theorems on integrability.

Theorem. *Let $f : [a, b] \to \mathbf{R}$ be continuous. Then f is Riemann integrable on $[a, b]$.*

Proof. We will verify the Cauchy condition. Let $\varepsilon > 0$ be given. Since f is continuous on the closed bounded interval $[a, b]$, f is uniformly continuous on $[a, b]$. Choose $\delta > 0$ as in the definition of uniform continuity of f on $[a, b]$ for the positive quantity $\frac{1}{2}\varepsilon/(b-a)$. Now let ${}^tP$ and ${}^tP'$ be any two tagged partitions of $[a, b]$ with $\|P\|$, $\|P'\| < \delta$. Let $Q = P \cup P'$ (or let Q be any common refinement of P and P'), and let ${}^tQ$ be an arbitrary tagging of Q. Let ${}^tP$ and ${}^tQ$ be labelled as in the preliminary computation. Notice that for any i between 1 and m, and any $j \in E_i$, s_i and t_j both belong to $[x_{i-1}, x_i]$. Since

$$|x_i - x_{i-1}| \leq \|P\| < \delta,$$

we have

$$|s_i - t_j| < \delta,$$

and hence

$$|f(s_i) - f(t_j)| < \frac{\varepsilon}{2(b-a)}.$$

Using the preliminary computation, we get

$$\begin{aligned}
\left|S(f, {}^tP) - S(f, {}^tQ)\right| &\leq \sum_{i=1}^{m} \sum_{j \in E_i} \left|f(s_i) - f(t_j)\right|(y_j - y_{j-1}) \\
&< \sum_{i=1}^{m} \sum_{j \in E_i} \frac{\varepsilon}{2(b-a)}(y_j - y_{j-1}) \\
&= \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{m} \sum_{j \in E_i} (y_j - y_{j-1}) \\
&= \frac{\varepsilon}{2(b-a)}(b-a) \\
&= \varepsilon/2.
\end{aligned}$$

Similarly,

$$\left|S(f, {}^tP') - S(f, {}^tQ)\right| < \varepsilon/2.$$

Combining these gives

$$\left|S(f, {}^tP) - S(f, {}^tP')\right| < \varepsilon. \qquad \blacksquare$$

Theorem. *Let $f : [a, b] \to \mathbf{R}$ be monotone. Then f is Riemann integrable on $[a, b]$.*

Proof. For definiteness we assume that f is increasing. Note that then $f(a) \leq f(b)$. Again, it suffices to check the Cauchy condition. Let $\varepsilon > 0$ be given. Choose

$$\delta = \frac{\varepsilon}{2\big(f(b) - f(a)\big) + 1}.$$

Let ${}^tP$ and ${}^tP'$ be any two tagged partitions of $[a, b]$ with $\|P\|$, $\|P'\| < \delta$, and let ${}^tQ$ be a common refinement, tagged arbitrarily. Let ${}^tP$ and ${}^tQ$ be labelled as in the preliminary computation. Notice that for any i between 1 and m, and any $j \in E_i$, s_i and t_j both belong to $[x_{i-1}, x_i]$. Since f is increasing,

$$\bigl|f(s_i) - f(t_j)\bigr| \le f(x_i) - f(x_{i-1}).$$

By the preliminary computation,

$$\begin{aligned}
\bigl|S(f, {}^tP) - S(f, {}^tQ)\bigr| &\le \sum_{i=1}^{m} \sum_{j \in E_i} \bigl|f(s_i) - f(t_j)\bigr|(y_j - y_{j-1}) \\
&\le \sum_{i=1}^{m} \sum_{j \in E_i} \bigl(f(x_i) - f(x_{i-1})\bigr)(y_j - y_{j-1}) \\
&= \sum_{i=1}^{m} \bigl(f(x_i) - f(x_{i-1})\bigr) \sum_{j \in E_i} (y_j - y_{j-1}) \\
&= \sum_{i=1}^{m} \bigl(f(x_i) - f(x_{i-1})\bigr)(x_i - x_{i-1}).
\end{aligned}$$

Using the fact that $x_i - x_{i-1} \le \|P\| < \delta$ for all i,

$$\begin{aligned}
&< \frac{\varepsilon}{2\bigl(f(b) - f(a)\bigr) + 1} \sum_{i=1}^{m} \bigl(f(x_i) - f(x_{i-1})\bigr) \\
&= \frac{\varepsilon}{2\bigl(f(b) - f(a)\bigr) + 1} \Bigl(f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})\Bigr) \\
&= \frac{\varepsilon}{2\bigl(f(b) - f(a)\bigr) + 1} \bigl(f(x_n) - f(x_0)\bigr) \\
&= \frac{\varepsilon}{2\bigl(f(b) - f(a)\bigr) + 1} \bigl(f(b) - f(a)\bigr) \\
&< \varepsilon/2.
\end{aligned}$$

Now the argument is finished just as in the proof of the previous theorem. ∎