Notes on Logarithms and Exponentials

The function $1/x$ does not have an antiderivative that can be expressed as a combination of known elementary functions (that is, in terms of polynomials, rational functions, radicals, trigonometric functions.) We use the fundamental theorem of calculus to define an antiderivative; it is called the *logarithm* function.

Definition. The *(natural) logarithm* function $\log : (0, \infty) \to \mathbf{R}$ is given by

$$\log x = \int_1^x \frac{1}{t}\, dt.$$

1. log is differentiable on $(0, \infty)$, and $\log'(x) = 1/x$.

2. $\log(1) = 0$.

3. log is strictly increasing on $(0, \infty)$, since $\log'$ is positive there.

4. (i) If $a > 0$ and $n \in \mathbf{Z}$, then $\log(a^n) = n \cdot \log(a)$.

Proof. Using the change of variable theorem with $\phi(x) = x^n$ gives

$$\begin{aligned} \log(a^n) &= \int_1^{a^n} \frac{1}{t}\, dt \\ &= \int_{\phi(1)}^{\phi(a)} \frac{1}{t}\, dt \\ &= \int_1^a \frac{1}{t^n} n t^{n-1}\, dt \\ &= n \int_1^a \frac{1}{t}\, dt \\ &= n \log(a). \end{aligned}$$

(ii) If $a > 0$ and $r \in \mathbf{Q}$, then $\log(a^r) = r \cdot \log(a)$.

Proof. Let $r = m/n$ with $m, n \in \mathbf{Z}$. Then

$$\begin{aligned} n \log(a^r) &= \log\big((a^r)^n\big), \text{ by (i)}, \\ &= \log(a^{rn}) \\ &= \log(a^m) \\ &= m \log a, \text{ by (i), again.} \end{aligned}$$

Therefore,

$$\log(a^r) = \frac{m}{n} \log a = r \log a.$$

5. By 2 and 3, $\log 2 > 0$. By 4, then, $\log(2^n) = n \cdot \log 2$ diverges to $+\infty$ as $n \to +\infty$, and diverges to $-\infty$ as $n \to -\infty$.

6. $\log : (0,\infty) \to \mathbf{R}$ is one-to-one and onto.

Proof. If $y \in \mathbf{R}$, choose $M > 0$ with $|y| < M$. By 5, we may choose $n \in \mathbf{J}$ such that $M < n \log 2$. Let $a = 2^{-n}$ and $b = 2^n$. Then $\log a < -M$ and $\log b > M$. By the Intermediate Value Theorem, there exists $x \in (a,b)$ with $y = \log x$. Thus log is onto. One-to-one follows from 3.

7. $\log(ab) = \log(a) + \log(b)$

Proof. Using the change of variable theorem with $\phi(x) = ax$ gives

$$\begin{aligned}\int_1^{ab} \frac{1}{t}\,dt &= \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \\ &= \int_1^a \frac{1}{t}\,dt + \int_{\phi(1)}^{\phi(b)} \frac{1}{t}\,dt \\ &= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{at}\,d(at) \\ &= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt.\end{aligned}$$

8. Since log is one-to-one and onto, it has an inverse function which we call $E : \mathbf{R} \to (0,\infty)$. The basic defining properties of E are:

$$\begin{aligned}E(\log(x)) &= x \text{ if } x \in (0,\infty) \\ \log(E(x)) &= x \text{ if } x \in \mathbf{R}.\end{aligned}$$

Moreover, by the inverse function theorem, E is differentiable, and

$$E'(\log(x)) = \frac{1}{\log'(x)} = \frac{1}{1/x} = x = E(\log(x)).$$

Thus $E' = E$. Since the range of E equals $(0,\infty)$, it follows that $E'(x) > 0$ for all $x \in \mathbf{R}$, and hence that E is strictly increasing.

9. (i) $E(rx) = E(x)^r$ for $r \in \mathbf{Q}$ and $x \in \mathbf{R}$.

Proof. Since log is onto, there is $a > 0$ with $x = \log a$. Then $a = E(x)$, and we have

$$\begin{aligned}E(rx) &= E(r \log a) \\ &= E\big(\log(a^r)\big), \text{ by 4(ii)}, \\ &= a^r, \text{ by 8}, \\ &= E(x)^r.\end{aligned}$$

(ii) $E(x+y) = E(x)E(y)$ for $x, y \in \mathbf{R}$.

Proof. Let $a, b > 0$ with $x = \log a$ and $y = \log b$. Then

$$\begin{aligned} E(x+y) &= E\bigl(\log a + \log b\bigr) \\ &= E\bigl(\log(ab)\bigr), \text{ by 7}, \\ &= ab, \text{ by 8}, \\ &= E(x)E(y). \end{aligned}$$

10. For any $a \in \mathbf{R}$ and $b \in (0, \infty)$ consider the expression b^a. Until now, this is not defined unless $a \in \mathbf{Q}$. But notice that if $a \in \mathbf{Q}$, then

$$\begin{aligned} b^a &= E(\log(b^a)), \text{ by 8}, \\ &= E(a\log(b)), \text{ by 4(ii)}. \end{aligned}$$

Now we *define* b^a to be $E(a\log(b))$ for *any* $a \in \mathbf{R}$. It is easy to check that the usual algebra of exponents still holds:

$$\begin{aligned} (b_1 b_2)^a &= b_1^a b_2^a \\ b^{a_1+a_2} &= b^{a_1} b^{a_2}. \end{aligned}$$

It is also easy to check the rules for differentiation of exponential functions:

$$\begin{aligned} \frac{d}{dx} x^b &= b x^{b-1} \\ \frac{d}{dx} b^x &= b^x \log(b). \end{aligned}$$

11. Now we define the number e by $e = E(1)$. Equivalently, e is defined by the equation $\log(e) = 1$, or $\int_1^e \frac{1}{t}\,dt = 1$. Then for any $x \in \mathbf{R}$,

$$E(x) = E(x \cdot 1) = E(x \cdot \log(e)) = e^x,$$

and in general,

$$b^a = E(a \log b) = e^{a \log b}.$$

12. Finally, we will prove the important formula: $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

Proof. First compute $\frac{d}{dt}\log(1+xt) = x/(1+xt)$. At $t = 0$ this gives x, so we have from the limit definition of the derivative,

$$x = \lim_{h\to 0} \frac{\log(1+xh)}{h}.$$

Letting $h = 1/n$, and letting $n \to \infty$, gives

$$x = \lim_{n\to\infty} n\log(1+x/n) = \lim_{n\to\infty} \log\bigl((1+x/n)^n\bigr).$$

Since e^x is a continuous function, it preserves limits (Theorem 3.1):

$$e^x = E(x) = \lim_{n\to\infty} E\Bigl(\log\bigl((1+x/n)^n\bigr)\Bigr) = \lim_{n\to\infty}(1+x/n)^n.$$

In particular, $e = \lim_{n\to\infty}(1+1/n)^n$.