MAT 578 **HOMEWORK 9** **Due: 10/25/07**

Solve three of the following problems.

37. Let $\{H_i : i \in I\}$ be a family of Hilbert spaces, and let $T_i \in B(H_i)$ for each i. Suppose that $\sup_{i\in I} \|T_i\| < \infty$, and let $T = \oplus_{i\in I} T_i \in B(\oplus_{i\in I} H_i)$.
    (a) Prove that T is invertible if and only if
        (i) T_i is invertible for all $i \in I$.
        (ii) $\sup_{i\in I} \|T_i^{-1}\| < \infty$.
    (b) Prove that $\sigma(T) \supseteq \overline{\cup_{i\in I} \sigma(T_i)}$.

38. Let $T \in B(H)$. Let $T = U|T|$ be the polar decomposition of T. Prove that T is normal if and only if U commutes with $|T|$ and U is normal.

39. Let T be an $n \times n$ diagonal complex matrix, with diagonal entries $\lambda_1, \ldots, \lambda_n$, thought of as a bounded linear operator on the Hilbert space $\mathbb{C}^n$. Prove that T is cyclic if and only if $\lambda_i \neq \lambda_j$ whenever $i \neq j$.

40. Let H be a Hilbert space, and let $M \subseteq H$ be a (closed) subspace.
    (a) Let $T \in B(H)$. Prove that $TM \subseteq M$ if and only if $TP_M = P_M T P_M$, where P_M denotes the orthogonal projection onto M. (We say that M is *invariant* under T if $TM \subseteq M$.)
    (b) Let $A \subseteq B(H)$ be a $*$-algebra of operators. Prove that M is invariant under A if and only if $P_M \in A'$ (where "invariant" here means that M is invariant under each operator in A). (Recall that the *commutant*, A', of A is defined by $A' = \{S \in B(H) : ST = TS \text{ for all } T \in A\}$.)
    (c) The $*$-algebra $A \subseteq B(H)$ is called *irreducible* if the only subspaces of H that are invariant under A are (0) and H. Prove that A is irreducible if and only if every nonzero vector in H is cyclic for A.
41. Let $A \subseteq B(H)$ be a $*$-algebra of operators.
    (a) Prove that A is irreducible if and only if $A' = \mathbb{C} \cdot 1_H$.
    (b) Suppose that A is irreducible and commutative. Prove that H is one-dimensional.