MAT 578 **HOMEWORK 8** **Due: 10/18/07**

Solve three of the following problems.

32. Let H be a Hilbert space and $W \in B(H)$ an invertible operator. Define $\theta : B(H) \to B(H)$ by $\theta(A) = WAW^{-1}$.
    (a) Prove that θ is an automorphism of the Banach algebra structure of $B(H)$.
    (b) Prove that θ is self-adjoint (i.e. $\theta(A^*) = \theta(A)^*$ for all $A \in B(H)$) if and only if W is a scalar multiple of a unitary operator.

33. Let $(X, \mathcal{M})$ be a measurable space, and let $f : X \to \mathbb{C}$ be a bounded Borel-measurable function. Let μ_1 and μ_2 be σ-finite measures on $\mathcal{M}$. For $i = 1, 2$, let T_i be the multiplication operator on $L^2(X, \mathcal{M}, \mu_i)$ defined by f. Suppose that μ_1 and μ_2 are mutually absolutely continuous. Prove that T_1 and T_2 are unitarily equivalent (i.e. there exists a unitary operator $W \in B\big(L^2(\mu_1), L^2(\mu_2)\big)$ such that $WT_1W^* = T_2$). (Hint: use the Radon-Nikodym theorem.)

34. Let A be a unital Banach algebra, and let $a \in A$. Define $f : \mathbb{C} \to A$ by $f(z) = e^{za}$. Prove that f is differentiable (in norm), and that $f'(z) = af(z)$.

35. In this problem you will prove the Fuglede theorem: if S and T commute, and T is normal, then S and T^* commute. (In fact, the proof works in any unital C^*-algebra. You won't actually use the underlying Hilbert space in the proof.)
    Let $S, T \in B(H)$ with T normal. Suppose that $ST = TS$.
    (a) Define $f : \mathbb{C} \to B(H)$ by $f(z) = e^{izT^*} S e^{-izT^*}$. Prove that f is entire, i.e. that f is differentiable in $\mathbb{C}$.
    (b) Let f be as in part (a). Prove that $f(z) = e^{i(zT^* + \bar{z}T)} S e^{-i(zT^* + \bar{z}T)}$. Use this to prove that $\|f(z)\| = \|S\|$ for all $z \in \mathbb{C}$. Use Liouville's theorem to conclude that f is constant.
    (c) Use the derivative of f from part (a) to prove that $ST^* = T^*S$.

36. Let $C(\mathbb{C})$ denote the (unital commutative $*$-) algebra of all continuous complex-valued functions on $\mathbb{C}$. For $X \subseteq \mathbb{C}$ compact, let $\|\cdot\|_X$ be the seminorm on $C(\mathbb{C})$ defined by $\|f\|_X = \sup\{|f(z)| : z \in X\}$.
    (a) Let $T \in B(H)$ be normal. Prove that there is a unique $*$-homomorphism $f \in C(\mathbb{C}) \mapsto f(T) \in B(H)$ such that $\zeta \mapsto T$ (where $\zeta(z) = z$), and such that $\|f(T)\| = \|f\|_{\sigma(T)}$ for all $f \in C(\mathbb{C})$.
    (b) Let $f \in C(\mathbb{C})$, and let $T_n \in B(H)$ be a norm-convergent sequence of normal operators, with $\lim_n T_n = T$. Prove that $\lim_n f(T_n) = f(T)$.