Solve three of the following problems.

27. Note: powers of the unilateral shift $S$, and of its adjoint, will be useful in this problem.
    (a) Prove that the adjoint map $T \to T^*$ on $B(H)$ is continuous for the weak operator topology, but not for the strong operator topology.
    (b) Prove that multiplication in $B(H)$ is strong operator continuous on bounded sets.

28. Prove that if $H$ is infinite dimensional, then multiplication in $B(H)$ is not strong operator continuous.

29. (a) Let $\mathcal{U}(H)$ be the set of unitary operators in $B(H)$. Prove that the strong operator closure of $\mathcal{U}(H)$ consists of isometries.
    (b) Let $S$ be the unilateral shift. Prove that $S \in \overline{\mathcal{U}(\ell^2)}^{\text{SOT}}$. (Hint: construct unitaries on $\ell^2\{1, \dots, n\}$ that strongly approximate $S$.)

30. Let $(X, \mu)$ be a $\sigma$-finite measure space, and let $\mathcal{B}(X)$ be the set of all bounded Borel functions $X \to \mathbb{C}$. Recall that for $f \in \mathcal{B}(X)$, the *essential range* of $f$ is the set of $\lambda \in \mathbb{C}$ such that $\mu\big(\{|f - \lambda| < \varepsilon\}\big) > 0$ for every $\varepsilon > 0$. If $f, g \in \mathcal{B}(X)$, we will write $f \sim g$ to mean that $f = g$ $\mu$-a.e. Prove that the essential range of $f$ equals
$$\bigcap \big\{\overline{g(X)} : g \in \mathcal{B}(X),\ g \sim f\big\}.$$

31. Let $A$ be a Banach algebra, and let $a_0, a_1, \dots, b_0, b_1, \dots \in A$. Suppose that $\sum_{n=0}^{\infty} \|a_n\| < \infty$ and $\sum_{n=0}^{\infty} \|b_n\| < \infty$. Let $x, y \in A$ be given by $x = \sum_{n=0}^{\infty} a_n$ and $y = \sum_{n=0}^{\infty} b_n$, and define $c_n \in A$ by $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Prove that $\sum_{n=0}^{\infty} \|c_n\| < \infty$, and that $xy = \sum_{n=0}^{\infty} c_n$.